On the large-scale geometry of mapping class groups

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Abstract

We provide two new routes for studying the geometry of mapping class groups, and of colourable hierarchically hyperbolic groups more generally.

Firstly, we show that they are quasimedian quasiisometric to finite-dimensional CAT(0) cube complexes, which are nonpositively-curved spaces with a particularly rich structure. Being quasimedian means that much of this additional structure is coarsely preserved, rather than just the metric.

Secondly, we show that they act geometrically on injective metric spaces. This lets us use facts from the theory of injective spaces to deduce properties of mapping class groups, such as semihyperbolicity.

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1 Introduction

It is generally accepted [CM82, Wus69] that the study of infinite discrete groups began in the 1880s with von Dyck's introduction of presentations as part of his studies on tesselations [Dyc82] and with Poincarȇ's treatment of Fuchsian groups [Poi82]. By then, Galois's work and permutation groups were being increasingly appreciated, Klein's Erlangen program was in full swing (indeed, free groups were discovered as examples of Fuchsian groups in hyperbolic geometry), and Lie's theory of continuous groups was in the air. However, it was Poincarȇ's introduction of the fundamental group for closed manifolds that really provided motivation for the subject [Poi95]: Tietze showed that such fundamental groups are always finitely presented [Tie08], and conversely it can be seen that in fact every finitely presented group is the fundamental group of a closed 4–manifold.

Much of the early theory was driven by Dehn's formulation [Deh10, Deh11] of the *word, conjugacy*, and *isomorphism problems* for finitely presented groups [CM82], but by the middle of the 20^{th}century, results such as the examples of Novikov and Boone [Nov55, Boo59] and the remarkable Adyan–Rabin theorem [Ady57, Rab58] had shown that even some of the most basic questions about finitely presented groups cannot be answered in full generality. It was therefore clear than any reasonable theory would need to make additional assumptions beyond finite presentability.

Success in this direction was soon found by building on some of Dehn's geometric ideas [LS77], and the perspective was crystallised by Gromov with his introduction of hyperbolic groups [Gro87]. The key insight is that if one has an action of a group on a metric space, then if the orbit maps are "approximate bijections", one can gain insights on the group by studying the geometry of the metric space. Gromov considered *δ–hyperbolic* metric spaces, citing a definition of Rips [Gro87, p.76], which can be characterised as geodesic spaces such that every geodesic triangle has a *δ–centre*: a point that is δ–close to all three sides. A group is hyperbolic if acts *geometrically* on some hyperbolic space. This "thin-triangle" condition behaves as a strong kind of negative curvature, and Gromov was able to prove a number of strong results on the structure of hyperbolic groups. Even better, there is a probabilistic sense in which almost all finitely presented groups are hyperbolic [Gro87, Ol'92].

However, there are many groups of independent interest that are not hyperbolic. Indeed, the strict negative curvature condition prevents $\mathbf{Z}^2$ from being a subgroup of any hyperbolic group [CDP90, Cor. 10.7.2], [GdlH90, Thm 8.34], which is rather restrictive. Given the success of hyperbolicity, it is natural to try to weaken its strict negative curvature to some kind of "nonpositive curvature": this has been a big industry in recent years.

There are two parts of the definition of hyperbolicity that can be tweaked. The first is to consider less representative actions on hyperbolic spaces, the acme being the theory of acylindrical hyperbolicity, which admits a number of distinct formulations [Osi16, DGO17, Sel97, Bow08, Ham08, BF02, Sis18]. The second, more flexible, approach is to consider other geometric conditions for the space being acted on. There are several notable instances of this, such as CAT(0) geometry [Bal95, BH99], relative hyperbolicity [Gro87, Far98, Bow12, Osi06, DS05], and semihyperbolicity [AB95], among others.

Another parallel approach to understanding groups that was particularly prevalent in the 80s and 90s is the theory of automaticity, which arose from conversations between Cannon and Thurston [Can84] and allows for efficient computer manipulations [ECH$^+$92, BGSS91, GS91a]. The definition involves a geometric part and an algorithmic part, but one can also consider spaces just satisfying geometric properties of this flavour [Alo92, Sho90], semihyperbolicity being one such property. The strongest form of automaticity is *biautomaticity* [GS91b], which approximately asks for a single generating set witnessing both automaticity and (for the geometric part) semihyperbolicity.

1.1 Mapping class groups

One key family of groups one would like to understand by nonpositive curvature techniques is that of the *mapping class groups* of (compact, orientable) surfaces. The mapping class group of a surface S is the group $\operatorname{MCG} S$ of isotopy classes of homeomorphisms of S (see item 3.18 for more discussion). These classically-studied groups already appeared in the work of Dehn and others a century ago [Deh22], and have been fundamental to the study of 3–manifolds since the geometric insights of Thurston [Thu88]. All but the simplest mapping class groups have free-abelian "Dehn-twist" subgroups, and thus are not hyperbolic.

1.1 Guiding question. Which forms of nonpositive curvature are satisfied by mapping class groups?

In general, mapping class groups are neither CAT(0) [KL96] nor relatively hyperbolic [AAS07, Bow05, KN04, BDM09]. Mosher proved that they are automatic [Mos95]. They are acylindrically hyperbolic by virtue of their natural (but not "approximately injective") actions on the *curve graphs*, which we describe in the next paragraph. It has recently been shown that mapping class groups are semihyperbolic (Definition 8.4) [HHP20, DMS20]; the main thrust of this thesis is to give a complete proof of this.

The curve graph of a surface S was introduced by Harvey [Har81]. Denoted $\mathcal{C}S$, it has a vertex for each isotopy class of simple closed curves on S, and two vertices are joined by an edge if the corresponding classes admit disjoint representatives. An important breakthrough was achieved by Masur–Minsky, who showed that $\mathcal{C}S$ is always a hyperbolic space [MM99]. This has been the starting point for the majority of the subsequent approaches to the geometry of $\operatorname{MCG} S$, such as the follow-up paper [MM00], in which it is shown that $\operatorname{MCG} S$ has some striking features of "piecewise hyperbolicity": it can be understood as being built from (products of) certain hyperbolic subspaces.

1.2 Remark. It should be pointed out that $\operatorname{MCG} S$ is sometimes called the *extended* mapping class group, with mapping class groups being defined as the index-2 subgroups $\operatorname{MCG}^+ S < \operatorname{MCG} S$ of *orientation-preserving* homeomorphisms [FM12]. However, all of the "piecewise hyperbolicity" is exhibited by $\operatorname{MCG} S$ as well, and moreover it is $\operatorname{MCG} S$ that appears in Ivanov's theorem [Iva97]. Because it is often easier to pass to finite-index subgroups than finite-index overgroups, $\operatorname{MCG} S$ appears to be the more natural object in this setting.

1.2 Other key players

We have already defined and discussed the importance of hyperbolic spaces. To reiterate the strength of the negative curvature resulting from asking *every* geodesic triangle to have a coarse centre, let us mention the *Morse lemma*. This states that every *quasigeodesic* (Definition 2.3) lies Hausdorff-close to a geodesic with the same endpoints, with the distance being bounded only by the hyperbolicity and quasigeodesic constants (see [Shc13, GS19] for optimal explicit constants). Compare this to the situation in, say, the Euclidean plane.

A large subclass of CAT(0) spaces that has attracted a lot of attention recently, in part thanks to its role in some major advances in 3–manifold theory [Ago13, Wis21], is that of *CAT(0) cube complexes* (Definition 2.6). These were introduced to geometric group theory by Gromov [Gro87], but equivalent objects have a longer history in graph theory and computer science [BK47, Win82]. They are simply connected cell complexes built out of cubes and satisfying a local nonpositive curvature condition that is very robust, in the sense that it arises naturally in several metric settings [Che00, Lea13, Mie14]. The more natural metric on a CAT(0) cube complex is the piecewise-ℓ^1–metric, and in this light they can be characterised as having a *median structure*, whereby every triple of points has a *median*: a point μ such that each pair can be joined by a geodesic passing through μ. More detail is given in Section 2.2.

A more recent and more general notion is that of *coarse median spaces* (Definition 2.11), introduced by Bowditch [Bow13]. These have a *coarse median map* that, on finite subsets, behaves like the median on a CAT(0) cube complex, up to some bounded error. Mapping class groups have such coarse median maps [BM11], and indeed this was largely the motivation for the introduction of coarse median spaces. Coarse median spaces are covered more fully in Section 2.3. The natural structure-preserving maps between coarse median spaces are known as *quasimedian* maps.

The "piecewise hyperbolicity" observed in mapping class groups by Masur–Minsky (and developed further in [Beh06, BKMM12]) has since been found in other groups, including all known groups acting geometrically on CAT(0) cube complexes [BHS17b, HS20] and most 3–manifold groups [BHS19, HRSS22]. The structure was formalised as *hierarchical hyperbolicity* (Section 3.1) by Behrstock–Hagen–Sisto [BHS17b, BHS19]. As it is rather unwieldy, the definition is necessarily quite complicated—although the technicalities of the structure are not a point of focus for us, hierarchical hyperbolicity does provide the backdrop for the arguments of Section 3, which is the bottleneck in generality for the thesis. See Section 1.4 for more discussion of the generality of the arguments herein.

Whilst semihyperbolicity is our end goal, it will not play any real role in the main arguments, and it only reappears in Section 8. As a matter of fact, my opinion is that semihyperbolicity is really just a curiosity, and I consider the results on *quasicubicality* (Definition 4.13) and *coarse injectivity* (Definition 6.16), as well as the general medians-first perspective, to be more interesting and useful. These will be discussed in the next subsection.

There is one more family of metric spaces that have not been mentioned yet but which play a crucial role: *injective* metric spaces (Definition 7.1). These have been studied since the 50s [AP56], but most of the work concerning them has focused on finite metric spaces. Whilst they did not make an appearance in geometric group theory till work of Lang [Lan13], they are becoming increasingly popular [CCG+20, HO21, Hae21]. In many respects, injective spaces behave like "an ℓ^∞ version of CAT(0) spaces"; see Section 7.

1.3 Main results and outline

Section 2 covers background that will be used throughout the thesis. It also contains a couple of important facts about coarse medians (Definition 2.11), most notably Proposition 2.13, which comes from [HP19]. It states that quasimedian quasiisometric embeddings (Definition 2.12) in CAT(0) cube complexes always give rise to quasimedian *quasiisometries* to CAT(0) cube complexes. This proposition is the mechanism by which we bring together what could be considered the first half of the thesis, namely Theorem 4.20, discussed below. The idea (essentially due to Bowditch) is that "any subspace of a CAT(0) cube complex that is approximately closed under taking medians is approximately a subspace closed under taking medians." This means that the image of any quasimedian quasiisometric embedding is Hausdorff-close to a CAT(0) subcomplex.

Section 3 is based on joint work with Mark Hagen [HP19]. Its purpose is to show that every *colourable* hierarchically hyperbolic group (HHG) admits a quasimedian quasiisometric embedding in a finite product of hyperbolic spaces; this is Theorem 3.27. The hyperbolic spaces in question are built via the Bestvina–Bromberg–Fujiwara construction, as in [BBF15] (see Section 3.2), and the embedding is (after passing to a finite-index subgroup) just an orbit map. The proof that it is a quasiisometric embedding uses the *distance formula*, similarly to [BBF15]. The main difficulty is to show that it is quasimedian. For this, we show that hierarchy paths get sent to unparametrised quasigeodesics in each factor, by decomposing them according to when they make progress in *relevant domains*. We then consider a hierarchy path triangle passing through the median, whose image is an unparametrised quasigeodesic triangle, and hence close to the hyperbolic median by the Morse lemma.

In Section 4, which is based on part of [Pet21], we consider a construction of Buyalo–Dranishnikov–Schroeder for embedding hyperbolic spaces in finite products of trees. The main technical result is Proposition 4.12, which states that the embedding of [BDS07] is quasimedian whenever it exists. This is done by a careful analysis of the construction, and by using certain facts from Section 2.3 showing that medians in hyperbolic spaces are fairly rigid. We can then apply Proposition 2.13. As a consequence of this and work of Mackay–Sisto [MS13] and Wright [Wri12], we obtain:

4.19 Theorem. *If X is a hyperbolic space, then X is quasimedian quasiisometric to a finite-dimensional CAT(0) cube complex if and only if X has finite asymptotic dimension.*

This generalises a result of Haglund–Wise [HW12, Thm 1.8]. Combining Theorem 4.19 with Theorem 3.27 and again using Proposition 2.13 yields:

4.20 Theorem. *Every colourable hierarchically hyperbolic group is quasimedian quasiisometric to a finite-dimensional CAT(0) cube complex.*

This applies in particular to mapping class groups, and in Section 4.4, we observe that the cube complexes produced must have some interesting properties. Theorem 4.20 is the main statement that we carry forward to the remaining sections. To tighten things up, we say that a coarse median space is *quasicubical* if it is quasimedian quasiisometric to a finite-dimensional CAT(0) cube complex.

In Section 5 we derive a "coarse Helly" property for *median-quasiconvex* subsets of quasicubical coarse median spaces (Corollary 5.6), which implies that median-quasiconvex subgroups have *bounded packing*. It is straightforward from the Helly property for convex subcomplexes of CAT(0) cube complexes, but it is important for the conclusion of Section 6. A related statement appears in [HHP20].

Section 6 is based on joint work with Thomas Haettel and Nima Hoda [HHP20]. In it, we modify an idea of Bowditch [Bow20b] to construct a new metric σ on coarse median spaces. This metric is analogous to the piecewise-ℓ^∞–metric on a CAT(0) cube complex, and the definition can be thought of as mimicking the characterisation of that as the length of the longest *chain of hyperplanes* separating two points. The section is devoted to showing that σ has certain nice properties when the coarse median space is quasicubical (and a couple of other small assumptions hold), the culmination being Theorem 6.20.

6.20 Theorem. *If (X,μ,d) is a locally finite, quasicubical,* roughly geodesic *coarse median space, then it is quasiisometric to the coarsely injective space (X,σ). Moreover, median-preserving isometries of (X,μ,d) are isometries of (X,σ).*

In the case where the space is a colourable HHG, the upshot of this is that it acts geometrically on a coarsely injective space. The most difficult part is the coarse injectivity. We break it down into two components. Firstly, we show that balls in (X,σ) are uniformly median-quasiconvex (Lemma 6.18). Secondly, we show that σ is *weakly roughly geodesic* (Proposition 6.15), which implies that if one has two metric balls such that the sum of the radii is greater than the distance between the centres, then they actually come close to each other. A detailed discussion of the argument for this is given in item 6.11. These two facts allow us to apply Corollary 5.6 to any family of balls, verifying coarse injectivity, Definition 6.16.

This is followed by Section 7, which considers injective metric spaces. A simple observation unifies Lang's construction of a bicombing [Lan13] and the barycentres obtained by Descombes [Des16]. We discuss a number of metric consequences of these constructions. Whilst the results of this section are not new in themselves, the unified presentation is.

Finally, in Section 8, we draw together the threads of the previous sections. As observed by Chalopin–Chepoi–Genevois–Hirai–Osajda [CCG$^+$20], any group acting geometrically on a coarsely injective space also acts geometrically on an injective space. By Theorems 6.20 and 4.20, this includes all colourable HHGs.

8.1 Theorem. *Every colourable HHG acts geometrically on some coarsely injective space.*

Using this, we obtain properties of colourable HHGs from the facts about injective spaces in Section 7; see Corollary 8.2. Semihyperbolicity is one such property. Since mapping class groups are colourable HHGs, this applies to them.

1.4 Discussion of generality

The various sections of this thesis have been kept as self-contained as possible, often with only one key result reappearing later. For this reason, their logical settings are not all the same. To summarise:

- Section 3 ([HP19]) deals with colourable HHGs. The main statement is Theorem 3.27.
- The majority of Section 4 ([Pet21]) is about bounded metric spaces. Its results are combined with Theorem 3.27 in Section 4.3 to obtain Theorem 4.20, which says that colourable HHGs are quasicubical.
- Section 5 concerns quasicubical coarse median spaces. It produces Corollary 5.6.
- Section 6 ([HHP20]) constructs a new metric for quasicubical coarse median spaces; it is necessary to assume local finiteness and rough geodesicity in certain places. Corollary 5.6 is used to prove Theorem 6.20, which states properties of the new metric.
- Section 7 treats injective metric spaces, ending with Theorem 7.28.
- Section 8 combines the results of the previous sections. Theorem 4.20 lets us apply Theorem 6.20 to colourable HHGs, yielding Theorem 8.1. We obtain consequences from Theorem 7.28.

Semihyperbolicity of mapping class groups was proved simultaneously and independently in [HHP20] and [DMS20]. The arguments are rather different. In [DMS20], it is deduced from a strong stability result for hulls of finite subsets of colourable HHGs. The arguments of this thesis are the same as in [HHP20], except we prove and use quasicubicality for colourable HHGs, whereas [HHP20] uses the property of having *quasicubical intervals*, which holds for all HHSs [BHS21, Thm 2.1], not just those that are colourable.

There are a few reasons why I have chosen not to work in the setting of coarse median spaces with quasicubical intervals in Section 6. Firstly, all the main known examples of HHGs are colourable, so the loss in generality does not seem to be substantial. Secondly, assuming quasicubicality makes a few arguments easier to conceptualise. Thirdly, I wanted to foreground the role of the coarse median. Finally, I wanted to minimise inertia: note that, except for a few basic facts about CAT(0) cube complexes, no external input is needed for the proof of Theorem 6.20. By comparison, the fact that mapping class groups have quasicubical intervals is a theorem of Behrstock–Hagen–Sisto [BHS21, Thm 2.1] (also see [Bow18, Thm 1.3]); it is proved by applying Sageev's construction to a well-chosen collection of walls. (The main result of [DMS20] strengthens it.)

2 Preliminaries

This section contains background material that underpins the thesis.

2.1 Metric notions and notations

For a point a and a subset A in a metric space (X, d), we write $B_X(a, r)$ for the open r–ball centred on a, and $\mathcal{N}_X(A, r)$ for the open r–neighbourhood about A. We write $\operatorname{diam} A = \sup\{\mathsf{d}(a, a') : a, a' \in A\}$. If $A, B \subset X$, then $\mathsf{d}(A, B) = \inf\{\mathsf{d}(a, b) : a \in A,\ b \in B\}$.

In geometric group theory, we are interested in the large-scale geometry of metric spaces. From this viewpoint, the spaces $\mathbf{Z}$ and $\mathbf{R} \times [0, n]$, for example, have "the same shape", and so should be considered equivalent. This is best captured by quasiisometries, and for groups by proper cobounded actions, which are the geometric actions alluded to in the introduction.

2.1 Definition (Quasiisometry)**.** For $\lambda \geqslant 1$, a map $f : X \to Y$ of metric spaces is a *λ–quasiisometric embedding* if

$$\frac{1}{\lambda} \mathsf{d}_X(x, y) - \lambda \ \leqslant\ \mathsf{d}_Y(f(x), f(y)) \ \leqslant\ \lambda\, \mathsf{d}_X(x, y) + \lambda$$

for all $x, y \in X$. It is *r–coarsely onto* if $f(X)$ is *r–coarsely dense*, in the sense that $Y = \mathcal{N}_Y(f(X), r)$. The map f is a *λ–quasiisometry* if it is a λ–coarsely onto λ–quasiisometric embedding.

2.2 Definition (Proper, cobounded)**.** An (isometric) action of a group G on a metric space is *proper* if for every ball B, only finitely many G–translates of B intersect B. An action is *cobounded* if some orbit is coarsely dense.

Quasiisometric embeddings are coarsely nonsingular versions of *coarsely Lipschitz* maps. We say that $f : X \to Y$ is (λ_1, λ_2)–coarsely Lipschitz if $\mathsf{d}_Y(f(x), f(x')) \leqslant \lambda_1\, \mathsf{d}_X(x, x') + \lambda_2$ for all $x, x' \in X$.

In the same way that geodesics are fundamental in the fine geometry of metric spaces, *quasigeodesics* are central to large-scale geometry.

2.3 Definition (Quasigeodesic)**.** A *λ–quasigeodesic* in a metric space X is a λ–quasiisometric embedding $\gamma : I \to X$, where $I \subset \mathbf{R}$ is a closed interval. An *unparametrised λ–quasigeodesic* is a subset of X that is the image of some λ–quasigeodesic.

One more notion that will appear in Section 3 and be used in Section 4 is that of *asymptotic dimension.* Asymptotic dimension was introduced by Gromov in [Gro93] as a group invariant; a nice survey can be found in [BD08]. We shall never actually engage with what asymptotic dimension is—it will be a technical tool for us only. A definition has been included for the sake of completeness.

2.4 Definition (Asymptotic dimension)**.** A family $\mathcal{U}$ of subsets of X is *r–disjoint* if $\mathsf{d}(U, U') > r$ for all $U, U' \in \mathcal{U}$, and is *uniformly bounded* if there is some R such that $\operatorname{diam} U \leqslant R$ for every $U \in \mathcal{U}$. We say that X has *asymptotic dimension at most n*, writing $\operatorname{asdim} X \leqslant n$, if for all r there are r–disjoint, uniformly bounded families $\mathcal{U}_0, \dots, \mathcal{U}_n$ such that $\bigcup_{i=0}^n \mathcal{U}_i$ is a cover of X.

2.2 CAT(0) cube complexes

Here we give a brief summary of some of the now-extensive theory of CAT(0) cube complexes; more thorough discussions can be found in [Hag12, Wis21], for example.

The following is a classical notion in graph theory [BK47, Ava61, BH83].

2.5 Definition (Median graph). A graph is *median* if for any vertices v_1, v_2, v_3 there is a unique vertex μ such that $\mathsf{d}(v_i, \mu) + \mathsf{d}(\mu, v_j) = \mathsf{d}(v_i, v_j)$ for each $i \neq j$.

Note that the ternary operation μ is symmetric and is 1–Lipschitz in each of its three coordinates.

The following is not the standard definition of CAT(0) cube complexes, but it is equivalent by [Che00, Thm 6.1].

2.6 Definition (CAT(0) cube complex). A *CAT(0) cube complex* is the space obtained from a median graph by attaching, in the obvious way, a unique cube $[0,1]^n$ to every subgraph isometric to a product of n edges, for every $n \geqslant 2$.

2.7 Metrics. There are three commonly considered metrics on CAT(0) cube complexes: the piecewise-ℓ^p–metrics d^p, for $p \in \{1, 2, \infty\}$. The metric d^2 is also called the CAT(0) metric, for if Q is a CAT(0) cube complex then (Q, d^2) is a CAT(0) space [Lea13]. The metric d^∞ is also natural because a cube complex Q is a CAT(0) cube complex if and only if (Q, d^∞) is an injective space (Definition 7.1) [Mie14]. It is the metric d^1 that best captures the combinatorics of a CAT(0) cube complex, though, for in this metric it is a median space and its 1–skeleton is isometrically embedded. One can also consider d^p for $p \in (1, \infty)$, which is always *Busemann-convex* [HHP].

These metrics are bilipschitz to one another for finite-dimensional CAT(0) cube complexes. Indeed, if $\dim Q < \infty$ and $p < q$, then

$$\mathsf{d}^\infty(x,y) \;\leqslant\; \mathsf{d}^q(x,y) \;\leqslant\; \mathsf{d}^p(x,y) \;\leqslant\; \dim Q\, \mathsf{d}^\infty(x,y).$$

Unless otherwise stated, all CAT(0) cube complexes will be considered with the metric d^1.

Formally, everything that we do with CAT(0) cube complexes could be done in the language of median graphs. In fact, as we are primarily concerned with medians, we only need the vertices, so we shall frequently identify a CAT(0) cube complex with its 0–skeleton without comment. Similarly, we shall count the vertex set of a graph as being a geodesic space. The chief advantage of the cube terminology is that it makes certain concepts more intuitive. For example, the *dimension* of a CAT(0) cube complex is the supremal dimension of its cubes. Another example is the concept of *hyperplanes.*

2.8 Definition (Hyperplane). A *midcube* is obtained from a cube by restricting one factor to $\{\frac{1}{2}\}$. Say that two midcubes in a CAT(0) cube complex are equivalent if they meet in a face, and extend the relation transitively. A *hyperplane* is an equivalence class of midcubes. See Figure 1.

The use of hyperplanes in studying CAT(0) cube complexes was pioneered by Sageev [Sag95], but the equivalent *Djoković relation* on edges of median graphs was known much earlier [Djo73]. Again, cubes make things more intuitive.

Each hyperplane h cuts the CAT(0) cube complex Q into two *halfspaces* h^-, h^+, the components of $Q \smallsetminus h$. We say that h *separates* the points of h^+ from those of h^-. Halfspaces are convex in the sense that $\mu(x, y, y') \in h^+$ whenever $y, y' \in h^+$, and similarly for h^-. Every subcomplex Y of a CAT(0) cube complex Q is contained in a unique smallest convex

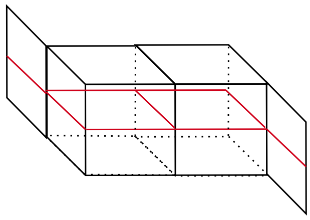

Figure 1: An example of a hyperplane in a CAT(0) cube complex.

subcomplex of Q, denoted $\operatorname{hull} Y$. In the case that $Y = \{x, y\}$ consists of only two points, $\operatorname{hull} Y$ is called the interval from x to y. In this case we have

$$\begin{aligned} \operatorname{hull} Y &= [x,y] = \{z \in Q : z \text{ lies on a geodesic from } x \text{ to } y\} \\ &= \{\mu(x,y,z) : z \in Q\}. \end{aligned}$$

2.9 Definition (Chain)**.** A *chain* of hyperplanes is a sequence $(h_i)_{i \in I \cap \mathbf{Z}}$, for some interval I, such that h_i separates h_{i-1} from h_{i+1} for all i.

A subset Y of a metric space X is said to be r–coarsely connected if for any $y, y' \in Y$ there is a sequence $y = y_0, y_1, \ldots, y_n = y'$ in Y with $\mathsf{d}(y_{i-1}, y_i) \leqslant r$ for all i. The following proposition is a combination of Proposition 2.8 and Lemma 2.11 from [HP19], the former of which is based on [Bow18, Prop. 4.1] (or [Fio21, Prop. 4.1]).

2.10 Proposition. *For all r, ν, and m, there exists k such that the following holds. Suppose that Y is an r–coarsely connected subcomplex of a ν–dimensional CAT(0) cube complex Q, and that Y has the property that $\mathsf{d}(\mu(y_1, y_2, y_3), Y) \leqslant m$ for all $y_1, y_2, y_3 \in Y$. There is an isometrically embedded (in the metric d^1, see item 2.7) CAT(0) subcomplex $Z \subset Q$ with Hausdorff-distance $\mathsf{d}_{Haus}(Y, Z) \leqslant k$.*

Proof. Decompose Y as a disjoint union of maximal 1–coarsely connected subsets Y_i. For each i, let S_i be the set of all j with $\mathsf{d}(Y_i, Y_j) \leqslant r$. Let Y' be the 1–coarsely connected subcomplex obtained from Y by adding, for each i, a geodesic of length at most r from Y_i to Y_j for each $j \in S_i$. Clearly $\mathsf{d}_{Haus}(Y, Y') \leqslant r$. Since μ is 1–Lipschitz in each factor, Y' has the property that $\mathsf{d}(\mu(y'_1, y'_2, y'_3), Y') \leqslant m + 3r$ for all $y'_1, y'_2, y'_3 \in Y'$. Thus, [Bow18, Prop. 4.1] provides a 1–coarsely connected induced median subgraph $Z' \subset Q$ that is at bounded Hausdorff distance from Y'.

It remains to show that Z' is isometrically embedded in Q, for then we can take Z to be the CAT(0) cube complex with 1–skeleton Z'. We show by induction that any two vertices $z, z' \in Z'$ can be joined by a geodesic of Q contained in Z'. This is clear if $\mathsf{d}(z, z') = 1$. If $\mathsf{d}(z, z') = n > 1$, then let $\gamma = (z = z_0, z_1, \ldots, z_m = z')$ be an edge path in Z' from z to z'. The edge path $\gamma' = \mu(z, z', \gamma)$ is a subset of $Z' \cap [z, z']$. There exists i such that $\mathsf{d}(\gamma'(i), z) = n-1$. As $\gamma'(i) \in [z, z']$, it is adjacent to z' in Q, hence in the induced subgraph Z'. By induction, there is a Q–geodesic from z to z' via $\gamma'(i)$. □

2.3 Coarse median spaces

Coarse median spaces were introduced by Bowditch [Bow13], and the class includes many examples of interest, such as mapping class groups, hyperbolic spaces, Teichmüller space with either of the usual metrics, CAT(0) cube complexes, and hierarchically hyperbolic spaces [Bow13, NWZ19, Bow16a, Bow20a, BHS19]. The motivation came from a construction of Behrstock–Minsky for mapping class groups [BM11] (see item 3.18), whose proof of the rapid decay property can be generalised to groups that are coarse median spaces [Bow14a].

2.11 Definition (Coarse median space)**.** A metric space (X, d) is a *coarse median space* if there is a map $\mu : X^3 \to X$ (which we refer to as the *coarse median*) and a function $\kappa : \mathcal{N}_0 \to \mathbf{R}$ such that the following conditions hold.

- μ is symmetric, and $\mu(x, x, y) = x$ for all $x, y \in X$.
- For any $x, x', y, y', z, z' \in X$ we have
$$\mathsf{d}(\mu(x, y, z), \mu(x', y', z')) \leqslant \kappa(0)(1 + \mathsf{d}(x, x') + \mathsf{d}(y, y') + \mathsf{d}(z, z')).$$
- For all $n \in \mathbf{N}$, if $A \subset X$ has cardinality at most n, then there is a finite CAT(0) cube complex Q with maps $f : A \to Q$ and $\bar{f} : Q \to X$ such that
  - $\mathsf{d}\left(\bar{f}\mu(v_1, v_2, v_3), \mu(\bar{f}(v_1), \bar{f}(v_2), \bar{f}(v_3))\right) \leqslant \kappa(n)$ for all $v_1, v_2, v_3 \in Q$;
  - $\mathsf{d}(a, \bar{f}f(a)) \leqslant \kappa(n)$ for all $a \in A$.

In Section 6, we shall write $\kappa(0) = \kappa_0$ to improve readability, and assume that $\kappa_0 \geqslant 1$ for convenience. We shall also write $\mu(x, y, z) = \mu_{xyz}$ when expedient.

The natural maps between coarse median spaces are *quasimedian* maps.

2.12 Definition (Quasimedian)**.** A map $f : X \to Y$ of coarse median spaces is λ–*quasimedian* if
$$\mathsf{d}_Y\left(f(\mu_{xyz}),\ \mu(fx, fy, fz)\right) \ \leqslant\ \lambda$$
for all $x, y, z \in X$.

The following proposition from [HP19] is a pivotal tool in this thesis; it will be used at a key point in Section 4.3. It shows that one can upgrade quasimedian quasiisometric embeddings in CAT(0) cube complexes to quasimedian quasiisometries.

2.13 Proposition. *If a coarsely connected coarse median space X admits a quasimedian quasiisometric embedding f in a finite-dimensional CAT(0) cube complex Q, then X is quasimedian quasiisometric to a CAT(0) cube complex.*

Proof. The image $f(X)$ is coarsely connected because f is coarsely Lipschitz, and $\mu(fx_1, fx_2, fx_3)$ is uniformly close to $f(X)$ for any $x_1, x_2, x_3 \in X$ because f is quasimedian. The result follows by applying Proposition 2.10 to $f(X)$. □

Let us finish this section with a few useful facts about coarse medians and hyperbolicity.

2.14 Proposition ([NWZ19, Thm 4.2])**.** *If X is a δ–hyperbolic space, then any coarse median on X is a bounded perturbation of a map sending each triple to a choice of δ–centre.*

2.15 Lemma. *If X and Y are hyperbolic spaces, then any map $f : X \to Y$ that sends geodesics to uniform unparametrised quasigeodesics is quasimedian. In particular, any quasiisometric embedding of hyperbolic spaces is quasimedian.*

Proof. According to Proposition 2.14, there is only a bounded ambiguity in the coarse median operations. Let $x_1, x_2, x_3 \in X$, and for each pair (i, j) let γ_{ij} be a uniform quasigeodesic from x_i to x_j that passes through $m = \mu_X(x_1, x_2, x_3)$. The image $f\gamma_{ij}$ is a uniform unparametrised quasigeodesic, so lies at bounded Hausdorff-distance from a geodesic with the same endpoints by the Morse lemma. Thus $f(m)$ is uniformly close to $\mu_Y(fx_1, fx_2, fx_3)$. □

Whilst it is the goal of nonpositive curvature to generalise hyperbolicity, there is some ambiguity as to what that should mean: hyperbolic spaces have many nice properties, and it is not clear exactly which ones should be considered indispensable. One can view Lemma 2.15 as saying that, at least in the world of coarse median spaces, it is quasimedian quasiisometries that should be considered as the fundamental maps, not mere quasiisometries, as it shows that there is no loss in adding the adjective in the hyperbolic setting.

The following is a kind of converse to Lemma 2.15.

2.16 Lemma. *Let X be a δ–hyperbolic space. If $\gamma : I \to X$ is λ–quasimedian and has r–coarsely connected image, then γ is an unparametrised $2(r+\delta+\lambda+1)$–quasigeodesic.*

Proof. Let $C = r + 2\delta + 2\lambda + 1$. After translating $\mathbf{R}$ we may assume that $0 \in I$. Let $t_0 = 0$. For $i > 0$, given t_{i-1}, let $t_i > t_{i-1}$ be minimal such that $\mathsf{d}(\gamma(t_{i-1}), \gamma(t_i)) \geqslant C$. For $i < 0$, given t_{i+1}, let $t_i < t_{i+1}$ be maximal such that $\mathsf{d}(\gamma(t_i), \gamma(t_{i+1})) \geqslant C$. We claim that $i \mapsto \gamma(t_i)$ is a quasigeodesic. Firstly, coarse connectivity shows that it is $(C+r)$–coarsely Lipschitz. In the other direction, let $i < k$ and let α be a geodesic from $\gamma(t_i)$ to $\gamma(t_k)$. By the triangle inequality, we have

$$\begin{aligned} \mathsf{d}(\gamma(t_i), \gamma(t_k)) &\geqslant \sum_{j=i}^{k-1} \big(\mathsf{d}(\gamma(t_j), \gamma(t_{j+1})) - \mathsf{d}(\gamma(t_j), \alpha) - \mathsf{d}(\gamma(t_{j+1}), \alpha)\big) \\ &\geqslant \sum_{j=i}^{k-1} \big(C - 2(\mathsf{d}(\mu_{\gamma(t_i), \gamma(t_j), \gamma(t_k)}, \alpha) + \lambda)\big) \\ &\geqslant \sum_{j=i}^{k-1} \big(C - 2(\delta + \lambda)\big) \;\geqslant\; k - i. \end{aligned}$$

□

Let X and Y be metric spaces equipped with n–ary operations $p_X : X^n \to X$ and $p_Y : Y^n \to Y$. We say that a map $f : X \to Y$ is a *coarse morphism* with respect to p_X and p_Y if there is a constant λ such that $\mathsf{d}_Y\big(fp_X(x_1, \dots, x_n), p_Y(fx_1, \dots, fx_n)\big) \leqslant \lambda$ for every $(x_1, \dots, x_n) \in X^n$.

2.17 Lemma. *Let X and Y be hyperbolic spaces. If a coarsely Lipschitz map $f : X \to Y$ is a coarse morphism with respect to the binary operations $\mu_X(\cdot, \cdot, x_0) : X^2 \to X$ and $\mu_Y(\cdot, \cdot, f(x_0)) : Y^2 \to Y$ for some $x_0 \in X$, then f is quasimedian.*

Proof. Let $x_1, x_2 \in X$, and let γ be a geodesic from x_1 to x_2. Let γ_i be a uniform quasigeodesic from x_0 to x_i that passes through $m = \mu_X(x_1, x_2, x_0)$. Let $\gamma_i' \subset \gamma_i$ be the subsegment from m to x_i. Then γ uniformly fellow-travels with the uniform quasigeodesic $\gamma_1' \cup \gamma_2'$.

The coarse morphism property of f tells us that the γ_i get mapped to uniform unparametrised quasigeodesics, and moreover that $f(m)$ is uniformly close to $\mu_Y(fx_1, fx_2, fx_0)$. In particular, the coarse intersection of $f\gamma_1'$ with $f\gamma_2'$ is uniformly bounded. This shows that $f\gamma_1' \cup f\gamma_2'$ is a uniform unparametrised quasigeodesic. Since γ fellow-travels with $\gamma_1' \cup \gamma_2'$, this implies that $f\gamma$ is a uniform unparametrised quasigeodesic.

We have shown that f sends geodesics to uniform unparametrised quasigeodesics, so we are done by Lemma 2.15. □

3 Colourable HHGs and projection complexes

This section is based on joint work with Mark Hagen [HP19]. The main result, Theorem 3.27, is that colourable hierarchically hyperbolic groups admit quasimedian quasiisometric embeddings in finite products of hyperbolic spaces.

The section is subdivided as follows. Section 3.1 covers necessary background and basic results on hierarchical hyperbolicity. Section 3.2 describes the Bestvina–Bromberg–Fujiwara construction, which is used to construct the relevant hyperbolic spaces. Section 3.3 contains the main result and its proof.

3.1 Hierarchical hyperbolicity

The central definitions of hierarchical hyperbolicity are rather technical, and in any case are not the best way to understand the area.

The concept came about because it was observed that certain disparate spaces and groups had a common structure that could be used for studying them [MM00, Hag14, KK14]. As many arguments about these examples only used certain key facts about this structure, it was propitious to abstract it as a framework that could be studied in its own right, especially since some natural arguments take place within this framework but outside the examples themselves. This led to a first definition of hierarchical hyperbolicity [BHS17b] in terms of those key facts.

Although this definition was well suited to proving facts about known examples, it suffered from the problem that it was difficult to verify in practice. To remedy this, the now-standard definition was introduced in [BHS19]; it consists of a list of individually simple properties that together imply the key facts. Since the actual geometry the definition is trying to capture already has several moving parts, and since these parts have been broken down into multiple pieces, the definition is necessarily quite complicated.

One should therefore take a dualistic approach. If one aims to prove that a space is hierarchically hyperbolic, then one should work in terms of the definition from [BHS19] (or use the sufficient conditions in [BHMS20]). However, if one is interested in working with spaces that are known to be hierarchically hyperbolic, then it is better to approach things from the plateau of the key facts of nature, rather than the thicket of the definition of artifice.

The reader can therefore safely skip the full definition of hierarchically hyperbolic spaces, which should really be viewed as a set of criteria to imply the *distance formula* below. It is quoted from [BHS19, Def. 1.1] for completeness. Discussion of the important points appears afterwards.

3.1 The definition from [BHS19]. "The q–quasigeodesic space (X, d_X) is a *hierarchically hyperbolic space* if there exists $\delta \geqslant 0$, an index set $\mathfrak{S}$, and a set $\{\mathcal{C}W : W \in \mathfrak{S}\}$ of δ–hyperbolic spaces $(\mathcal{C}U, \mathsf{d}_U)$, such that the following conditions are satisfied:

1. **(Projections.)** There is a set $\{\pi_W : X \to 2^{\mathcal{C}W} \mid W \in \mathfrak{S}\}$ of *projections* sending points in X to sets of diameter bounded by some $\xi \geqslant 0$ in the various $\mathcal{C}W \in \mathfrak{S}$. Moreover, there exists K so that for all $W \in \mathfrak{S}$, the coarse map π_W is (K, K)–coarsely Lipschitz and $\pi_W(X)$ is K–quasiconvex in $\mathcal{C}W$.
2. **(Nesting.)** $\mathfrak{S}$ is equipped with a partial order $\sqsubset$, and either $\mathfrak{S} = \varnothing$ or $\mathfrak{S}$ contains a unique $\sqsubset$–maximal element; when $V \sqsubset W$, we say V is *nested* in W. (We emphasize that $W \sqsubset W$ for all $W \in \mathfrak{S}$.) For each $W \in \mathfrak{S}$, we denote by $\mathfrak{S}_W$ the set of $V \in \mathfrak{S}$ such that $V \sqsubset W$. Moreover, for all $V, W \in \mathfrak{S}$ with $V \sqsubsetneq W$ there is a specified subset $\rho^V_W \subset \mathcal{C}W$ with $\operatorname{diam}_{\mathcal{C}W}(\rho^V_W) \leqslant \xi$. There is also a *projection* $\rho^W_V : \mathcal{C}W \to 2^{\mathcal{C}V}$.

(The similarity in notation is justified by viewing ρ^V_W as a coarsely constant map $\mathcal{C}V \to 2^{\mathcal{C}W}$.)

3. **(Orthogonality.)** $\mathfrak{S}$ has a symmetric and anti-reflexive relation called *orthogonality*: we write $V \perp W$ when V, W are orthogonal. Also, whenever $V \sqsubset W$ and $W \perp U$, we require that $V \perp U$. We require that for each $T \in \mathfrak{S}$ and each $U \in \mathfrak{S}_T$ for which $\{V \in \mathfrak{S}_T \mid V \perp U\} \neq \emptyset$, there exists $W \in \mathfrak{S}_T - \{T\}$, so that whenever $V \perp U$ and $V \sqsubset T$, we have $V \sqsubset W$. Finally, if $V \perp W$, then V, W are not $\sqsubset$–comparable.
4. **(Transversality and consistency.)** If $V, W \in \mathfrak{S}$ are not orthogonal and neither is nested in the other, then we say V, W are *transverse*, denoted $V \pitchfork W$. There exists $\kappa_0 \geqslant 0$ such that if $V \pitchfork W$, then there are sets $\rho^V_W \subseteq \mathcal{C}W$ and $\rho^W_V \subseteq \mathcal{C}V$ each of diameter at most ξ and satisfying:
$$\min\left\{\mathsf{d}_W(\pi_W(x), \rho^V_W), \mathsf{d}_V(\pi_V(x), \rho^W_V)\right\} \leqslant \kappa_0$$
for all $x \in X$.
For $V, W \in \mathfrak{S}$ satisfying $V \sqsubset W$ and for all $x \in X$, we have:
$$\min\left\{\mathsf{d}_W(\pi_W(x), \rho^V_W), \operatorname{diam}_{\mathcal{C}V}(\pi_V(x) \cup \rho^W_V(\pi_W(x)))\right\} \leqslant \kappa_0.$$
The preceding two inequalities are the *consistency inequalities* for points in X.
Finally, if $U \sqsubset V$, then $\mathsf{d}_W(\rho^U_W, \rho^V_W) \leqslant \kappa_0$ whenever $W \in \mathfrak{S}$ satisfies either $V \subsetneq W$ or $V \pitchfork W$ and $W \not\perp U$.
5. **(Finite complexity.)** There exists $n \geqslant 0$, the *complexity* of X (with respect to $\mathfrak{S}$), so that any set of pairwise–$\sqsubset$–comparable elements has cardinality at most n.
6. **(Large links.)** There exist $\lambda \geqslant 1$ and $E \geqslant \max\{\xi, \kappa_0\}$ such that the following holds. Let $W \in \mathfrak{S}$ and let $x, x' \in X$. Let $N = \lambda\, \mathsf{d}_W(\pi_W(x), \pi_W(x')) + \lambda$. Then there exists $\{T_i\}_{i=1,\dots,\lfloor N \rfloor} \subseteq \mathfrak{S}_W - \{W\}$ such that for all $T \in \mathfrak{S}_W - \{W\}$, either $T \in \mathfrak{S}_{T_i}$ for some i, or $\mathsf{d}_T(\pi_T(x), \pi_T(x')) < E$. Also, $\mathsf{d}_W(\pi_W(x), \rho^{T_i}_W) \leqslant N$ for each i.
7. **(Bounded geodesic image.)** There exists $E > 0$ such that for all $W \in \mathfrak{S}$, all $V \in \mathfrak{S}_W - \{W\}$, and all geodesics γ of $\mathcal{C}W$, either $\operatorname{diam}_{\mathcal{C}V}(\rho^W_V(\gamma)) \leqslant E$ or $\gamma \cap \mathcal{N}_E(\rho^V_W) \neq \emptyset$.
8. **(Partial Realization.)** There exists a constant α with the following property. Let $\{V_j\}$ be a family of pairwise orthogonal elements of $\mathfrak{S}$, and let $p_j \in \pi_{V_j}(X) \subseteq \mathcal{C}V_j$. Then there exists $x \in X$ so that:
   - $\mathsf{d}_{V_j}(x, p_j) \leqslant \alpha$ for all j,
   - for each j and each $V \in \mathfrak{S}$ with $V_j \sqsubset V$, we have $\mathsf{d}_V(x, \rho^{V_j}_V) \leqslant \alpha$, and
   - if $W \pitchfork V_j$ for some j, then $\mathsf{d}_W(x, \rho^{V_j}_W) \leqslant \alpha$.
9. **(Uniqueness.)** For each $\kappa \geqslant 0$, there exists $\theta_u = \theta_u(\kappa)$ such that if $x, y \in X$ and $\mathsf{d}_X(x, y) \geqslant \theta_u$, then there exists $V \in \mathfrak{S}$ such that $\mathsf{d}_V(x, y) \geqslant \kappa$."

3.2. A hierarchically hyperbolic space (HHS) is generally written as a pair $(X, \mathfrak{S})$, where X is a metric space and $\mathfrak{S}$ is a set, though it comes with more associated information, including a constant E such that the following hold.

- For each *domain* $U \in \mathfrak{S}$, there is a geodesic hyperbolic space $\mathcal{C}U$ and an (E, E)–coarsely Lipschitz, E–coarsely onto map $\pi_U : X \to \mathcal{C}U$.
- $\mathfrak{S}$ has three pairwise mutually exclusive relations: *nesting*, $\sqsubset$, a partial order; *orthogonality*, $\perp$, symmetric; and *transversality*, $\pitchfork$, their (symmetric) complement.
- If $U \pitchfork V$ or $U \subsetneq V$, then there is a specified point $\rho^U_V \in \mathcal{C}V$.
- If $U \pitchfork V$, then any $x \in X$ with $\mathsf{d}_{\mathcal{C}U}(\pi_U(x), \rho^V_U) > E$ has $\mathsf{d}_{\mathcal{C}V}(\pi_V(x), \rho^U_V) \leqslant E$. This is known as *consistency*.

3.3 Remark. In the definition of [BHS19], the maps π_U are only required to be quasiconvex, and the points ρ_V^U are allowed to be bounded sets. One can pass to the version stated here by increasing E.

We shall follow the convention of compressing notation by writing $\mathsf{d}_U(x,\cdot)$ for $\mathsf{d}_{\mathcal{C}U}(\pi_U(x),\cdot)$ when $x \in X$ and $U \in \mathfrak{S}$.

Let us now state two of the key facts that are part of the definition in [BHS17b] but are consequences of the definition in [BHS19]. For numbers r, s, let $\{\!\{r\}\!\}_s$ be equal to r if $r \geqslant s$ and 0 otherwise.

3.4 Distance formula. Suppose that $(X,\mathfrak{S})$ is an HHS. There is a constant $s_0 \geqslant 100E$ such that for any $s \geqslant s_0$ there exists A_s such that

$$\frac{1}{A_s}\sum_{U\in\mathfrak{S}}\{\!\{\mathsf{d}_U(x,y)\}\!\}_s - A_s \;\leqslant\; \mathsf{d}_X(x,y) \;\leqslant\; A_s\sum_{U\in\mathfrak{S}}\{\!\{\mathsf{d}_U(x,y)\}\!\}_s + A_s$$

holds for every $x, y \in X$.

3.5 Definition (Hierarchy path)**.** A *D–hierarchy path* in X is a D–quasigeodesic $\gamma \subset X$ such that $\pi_U\gamma$ is an unparametrised D–quasigeodesic for every $U \in \mathfrak{S}$.

3.6 Ubiquity of hierarchy paths. Suppose that $(X,\mathfrak{S})$ is an HHS. There is a constant D_0 such that every pair of points in X is joined by a D_0–hierarchy path.

There are a few additional pieces of structure that we shall need, the most fundamental of which is a coarse median. This could also be viewed as a defining fact, and indeed, in [Bow18], Bowditch uses a formulation that is similar to hierarchical hyperbolicity and includes the property of being a coarse median space as an axiom.

3.7 Proposition ([BHS19, Thm 7.3],[Bow13, Prop. 10.1])**.** *If $(X,\mathfrak{S})$ is an HHS, then there is a map $\mu : X^3 \to X$ making (X,μ) a coarse median space. Moreover, there is a coarsely unique choice of μ such that the π_U are uniformly quasimedian. That is, $\pi_U(\mu(x,y,z))$ is uniformly close to $\mu_{\mathcal{C}U}(\pi_U x, \pi_U y, \pi_U z)$ for any $x, y, z \in X$ and any $U \in \mathfrak{S}$.*

We fix a choice of μ such that the π_U are quasimedian. The following simple consequence of this choice of μ appears as the base case of [RST18, Prop. 5.6], and can also be deduced from [BHS21, Lem. 1.37, Thm 2.1], though the latter result involves more machinery.

3.8 Lemma. *Let $(X,\mathfrak{S})$ be an HHS. There is a constant D_1 such that for any $x_1, x_2, x_3 \in X$ there are D_1–hierarchy paths γ_{ij} from x_i to x_j passing through $\mu(x_1,x_2,x_3)$.*

Proof. Let γ_i be a D_0–hierarchy path from x_i to $m = \mu(x_1,x_2,x_3)$, and let γ_{ij} be the concatenation of γ_i with γ_j (reversed). Because $\pi_U(m)$ is uniformly close to a geodesic $[\pi_U(x_i),\pi_U(x_j)]$, we have that $\pi_U\gamma_{ij}$ is a uniform unparametrised quasigeodesic for every $U \in \mathfrak{S}$. To show that γ_{ij} is itself a uniform quasigeodesic, one applies the distance formula with a sufficiently large constant $s \geqslant 2s_0$, to find that $\mathsf{d}(\gamma_i(t_1),\gamma_j(t_2))$ is linearly approximated by $\mathsf{d}(\gamma_i(t_1),m) + \mathsf{d}(m,\gamma(t_2))$. □

3.9 Remark. There is an alternative definition of hierarchy paths that is more in keeping with the main themes of this thesis. Namely, it follows from Lemmas 2.15 and 2.16 and the coarse uniqueness of μ that a quasiisometric embedding $I \to X$ is a hierarchy path if and only if it is quasimedian. (Note that both the coarse median and the status of being a hierarchy path depend on the choice of HHS structure.) More generally, those lemmas show that a map $I \to X$ is an unparametrised hierarchy path if and only if it is quasimedian. We shall not make use of this version viewpoint.

As well as a hyperbolic space, there is a kind of maximal subspace of X associated with each $U \in \mathfrak{S}$, called the *product region* of U. Although these can (coarsely) be given a product structure [BHS19, §5.2], we shall only need the metric structure.

3.10 Definition (Product region)**.** The *product region* of $U \in \mathfrak{S}$ is the set $P_U = \{x \in X : \mathsf{d}(x, \rho^U_V) \leqslant E$ whenever $U \pitchfork V$ or $U \subsetneq V\}$.

3.11. As can be seen from the finiteness of the right-hand side of the distance formula, for any $x, y \in X$, there can be only finitely many $U \in \mathfrak{S}$ with $\mathsf{d}_U(x,y) \geqslant s_0$. For $r \geqslant 100E$, we write $\mathrm{Rel}_r(x,y)$ for the set of domains $U \in \mathfrak{S}$ that are *r–relevant* to (x,y), in the sense that $\mathsf{d}_U(x,y) \geqslant r$. There is a partial order on $\mathrm{Rel}_r(x,y)$ that restricts to a total order on every pairwise transverse subset. Namely, if $U, V \in \mathrm{Rel}_r(x,y)$ are transverse, then $U < V$ if $\mathsf{d}_V(x, \rho^U_V) \leqslant E$ [BHS19, Prop. 2.8]. In this case, consistency says that $\mathsf{d}_U(y, \rho^V_U) \leqslant E$.

3.12 Definition (Automorphism)**.** Let $C \geqslant 0$. A *C–HHS-automorphism* is an isometry $\phi : X \to X$ that preserves the HHS structure as follows.

- There is an induced bijection $\phi_{\mathfrak{S}} : \mathfrak{S} \to \mathfrak{S}$ that preserves the relations $\sqsubset, \perp, \pitchfork$.
- There is an isometry $\phi_U : \mathcal{C}U \to \mathcal{C}(\phi_{\mathfrak{S}}U)$ for each $U \in \mathfrak{S}$.
- For every $U \in \mathfrak{S}$, the maps $\phi_U \pi_U$ and $\pi_{\phi_{\mathfrak{S}}U}\phi$ commute up to an error of at most C.
- $\mathsf{d}\left(\phi_U(\rho^V_U),\, \rho^{\phi_{\mathfrak{S}}V}_{\phi_{\mathfrak{S}}U}\right) \leqslant C$ whenever $V \pitchfork U$ or $V \subsetneq U$.

We shall abuse notation slightly by writing π for both $\phi_{\mathfrak{S}}$ and ϕ_U. Note that the set of C–HHS-automorphisms need not form a group. Nonetheless, we say that a group G acts on $(X, \mathfrak{S})$ *by HHS automorphisms* if it acts by isometries and there is a constant C such that every $g \in G$ is a C–HHS-automorphism. It turns out that, given an action of G by HHS automorphisms, the HHS structure can always be perturbed so that every $g \in G$ is a 0–HHS-automorphism [DHS20, §2.1]. We shall therefore assume that this stronger condition holds whenever we have a group acting on an HHS by HHS automorphisms.

3.13 Definition (HHG)**.** A *hierarchically hyperbolic group* (HHG) is a group G with a choice of word metric and HHS structure $(G, \mathfrak{S})$ such that G acts on itself by HHS automorphisms, and cofinitely on $\mathfrak{S}$.

To recapitulate, if $(G, \mathfrak{S})$ is an HHG and $g, h \in G$, then there are isometries $g : \mathcal{C}U \to \mathcal{C}gU$ and $h : \mathcal{C}gU \to \mathcal{C}hgU$ such that $h \cdot g = hg : \mathcal{C}U \to \mathcal{C}hgU$. Moreover, $g\pi_U(x) = \pi_{gU}(gx)$ and $g\rho^U_V = \rho^{gU}_{gV}$.

3.14 Examples of HHGs.

- Mapping class groups of finite-type surfaces: this is a combination of [Har81, MM99, MM00, Beh06, BKMM12].
- All known cocompactly cubulated groups, by [HS20], using [Hag14, BHS17b].
- Artin–Tits groups of extra-large type [HMS21].
- The genus 2 handlebody group, by [Che20], using [HH21].
- Certain non–virtually-torsionfree lattices in products of trees [Hug21].
- Many 3–manifold groups, by [HRSS22], using [BHS19].
- Many quotients and extensions of (subgroups of) mapping class groups [BHS17a, BHMS20, DDLS20, DDLS21, Rus21].
- Many combinations of other HHGs [BHS19, BR20a, RS20, BR20b].

Note that the property of being an HHG does not pass to finite-index overgroups [PS20].

3.15 Remark. The regulatory conditions in the definition of an HHS automorphism imply that HHS automorphisms coarsely preserve the median μ of Proposition 3.7. We can make this exact when $(G,\mathfrak{S})$ is an HHG whose diagonal action on ${}^{G^3}\!/_{S_3}$ is free, that is when G has no 2– or 3–torsion. Indeed, for such G, pick a transversal T for the diagonal action of G on ${}^{G^3}\!/_{S_3}$. For every $x_1, x_2, x_3 \in G$ there is a unique $\{t_1, t_2, t_3\} \in T$ such that there is some $g \in G$ with $gt_i = x_i$ for all i. Define $\mu'(x_1, x_2, x_3) = g\mu(t_1, t_2, t_3)$. This map is G–equivariant, in the sense that $g\mu'(x,y,z) = \mu'(gx, gy, gz)$ for all $g, x, y, z \in G$, and it differs from μ by a bounded perturbation.

Otherwise μ' can fail to be symmetric, so we have to settle for coarse symmetry. We could instead allow μ to be bounded-set–valued and declare $\mu''(x_1, x_2, x_3) = \bigcup_{\sigma \in S_3} \mu'(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ (which fits with the formulation of coarse median structures in [Fio21]), but in general it is less disruptive to have coarse symmetry, as in the original definition of coarse median spaces [Bow13]. In any case, we never directly use symmetry of μ. By increasing E, we can and shall assume that μ is G–equivariant, meaning that G acts on itself by median-preserving isometries.

If one really wants symmetry and for medians to be points, then this can be achieved by relaxing the condition that the metric space underlying an HHG is exactly a Cayley graph, which is perfectly natural in the coarse context. There are a couple of options. Because symmetry only fails by a bounded amount, the simplest option is to just cone-off any non–symmetric medians and have the resulting space underlie the HHG, with the median being the cone-point. Alternatively, one can use an injective space on which G acts properly coboundedly, by [HHP20, Cor. 3.8, 3.10] (Theorem 8.1 and Lemma 7.7 in the colourable case), and define $\mu''(x_1, x_2, x_3)$ to be the barycentre of the $\mu'(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ (see Section 7.4).

There is one remaining result about HHGs that we shall use.

3.16 Theorem ([BHS17a, Cor. 3.3])**.** *If $(G,\mathfrak{S})$ is an HHG, then every $\mathcal{C}U$ has finite asymptotic dimension.*

We shall actually be working with a subset of HHGs, namely those with a *colouring*.

3.17 Definition (Colourable)**.** An HHG $(G,\mathfrak{S})$ is *colourable* if there is a partition $\mathfrak{S} = \bigsqcup_{i=1}^{\chi} \mathfrak{S}_i$ such that each $\mathfrak{S}_i$ consists of pairwise transverse domains, and G acts on $\{\mathfrak{S}_1, \dots, \mathfrak{S}_\chi\}$ by permutations. Equivalently, there is a finite-index subgroup $G^{\mathrm{chr}} \lhd G$ such that each $\mathfrak{S}_i$ is G^{chr}–invariant.

Not all HHGs are colourable; indeed Hagen has constructed an HHG that is an amalgamated product of colourable HHGs but is not itself colourable [Hag21]. However, all the key examples forming the basis of list 3.14 are colourable.

3.18 Mapping class groups. Hierarchical hyperbolicity is modelled on the structure enjoyed by mapping class groups. For a surface $S = S_{g,p}$ with genus g and p punctures, it is a classical theorem of Dehn that $\mathrm{MCG}\,S$ is finitely generated [Deh87]; in fact it is 2–generated if $g \geqslant 3$ [Mon21] or $p \leqslant 1$ [Kor05]. The spaces $\mathcal{C}U$ are (mostly) the curve graphs [Har81] of subsurfaces, which were shown to be unbounded and hyperbolic by Masur–Minsky [MM99]; there are now several articles proving that the hyperbolicity constant is independent of the surface [Aou13, Bow14b, CRS14, HPW15].

Consistency was proved by Behrstock [Beh06], and a number of other key facts appear in [BKMM12]. More discussion can be found in [CLM12]. The distance formula and results on hierarchy paths come from the original article by Masur–Minsky [MM00]. The coarse median was first constructed in [BM11]. Finiteness of the asymptotic dimensions of the $\mathcal{C}U$ was established by Bell–Fujiwara [BF08], using work of Bowditch [Bow08]. The bound was made explicit by Webb [Web15] and was later tightened by Bestvina–Bromberg [BB19]. Colourability is due to Bestvina–Bromberg–Fujiwara [BBF15].

3.19 Previous work. There has by now been a substantial amount of work on HHSs and HHGs, for instance providing more examples (and non-examples) than just those listed above [Vok17, Spr17, Hae20, PS20], considering notions of *boundaries* for HHSs [DHS17, Mou19, CDG20, NQ22], and establishing consequences of hierarchical hyperbolicity [BHS21, ABD21, RST18, ANS$^+$19, HHL20, Sel22].

3.2 The Bestvina–Bromberg–Fujiwara construction

The hyperbolic spaces in whose product we shall embed our colourable HHG are produced by a construction due to Bestvina–Bromberg–Fujiwara [BBF15]. A modified version of the construction was given in [BBFS20], but whilst this simplifies certain proofs about the construction itself, for us it would involve modifying the HHS structure a little, so we stick to the original construction.

Having said that, we replace certain bounded sets in the construction by points for convenience, as that is the situation we shall be operating in.

3.20 Projection axioms.

Let $\mathcal{Y}$ be a collection of geodesic metric spaces, with specified points $\rho_Y^X \in Y$ for every distinct $X, Y \in \mathcal{Y}$. We say that $\mathcal{Y}$ *satisfies the projection axioms with constant* θ if the following hold.

$$\text{If } X, Y, Z \text{ are distinct and } \mathsf{d}_Y(\rho_Y^X, \rho_Y^Z) > \theta, \text{ then } \mathsf{d}_Z(\rho_Z^X, \rho_Z^Y) \leqslant \theta. \tag{P1}$$

$$\text{For } X \neq Z, \text{ the set } \{Y : \mathsf{d}_Y(\rho_Y^X, \rho_Y^Z) > \theta\} \text{ is finite.} \tag{P2}$$

If, moreover, a group G acts on $\mathcal{Y}$, and each $g \in G$ induces isometries $g : Y \to gY$, then we say that the projection axioms are satisfied G*–equivariantly* if the isometries satisfy $g_1 g_2 = g_1 \cdot g_2$, and if $g\rho_Y^X = \rho_{gY}^{gX}$ holds for any $X, Y \in \mathcal{Y}$.

According to [BBF15], there is a constant $\Theta(\theta)$ such that the following holds for any $K \geqslant \Theta$ and some choice of L in terms of K and θ.

3.21 The quasitree of metric spaces. Given $\mathcal{Y}$ and θ, the quasitree of metric spaces $\mathcal{C}_K\mathcal{Y}$ is obtained from the disjoint union $\bigsqcup_{Y \in \mathcal{Y}} Y$ by attaching an edge of length $L(K, \theta)$ from ρ_Z^X to ρ_X^Z whenever $\mathsf{d}_Y^{\mathfrak{p}}(\rho_Y^X, \rho_Y^Z) \leqslant K$ for all $Y \in \mathcal{Y}$, where $\mathsf{d}_Y^{\mathfrak{p}}$ is a small perturbation of d_Y; see item 3.23. Note that if the projection axioms are satisfied G–equivariantly, then G acts on $\mathcal{C}_K\mathcal{Y}$.

The terminology *quasitree of metric spaces* comes from the fact that if the metric spaces in $\mathcal{Y}$ are uniformly bounded, then the resulting space is a quasitree. Recall that a quasitree is a geodesic space that is quasiisometric (equivalently *roughly isometric* [Ker20]) to a tree. The following is a combination of Theorems 4.17 and 4.24 of [BBF15].

3.22 Theorem ([BBF15])**.** *Suppose that $\mathcal{Y}$ contains finitely many isometry types of metric space. If every element of $\mathcal{Y}$ is hyperbolic, then $\mathcal{C}_K\mathcal{Y}$ is hyperbolic. If every element of $\mathcal{C}_K\mathcal{Y}$ has finite asymptotic dimension, then $\mathcal{C}_K\mathcal{Y}$ has finite asymptotic dimension.*

3.23. For $X, Y \in \mathcal{Y}$ and $x \in X$, define $\mathfrak{p}_Y(x) \in Y$ to be x if $Y = X$, and ρ_Y^X otherwise. According to [BBF15, Cor. 4.10], the map $\mathfrak{p}_Y$ coarsely agrees with the closest-point projection map to Y inside $\mathcal{C}_K\mathcal{Y}$.

For each $Y \in \mathcal{Y}$, the perturbed function $\mathsf{d}_Y^{\mathfrak{p}} : \mathcal{C}_K\mathcal{Y} \times \mathcal{C}_K\mathcal{Y} \to [0, \infty)$ is a symmetric map that sends the diagonal to 0 and, up to an error depending only on θ, satisfies the triangle inequality. By [BBF15, Thm 3.3], there is a constant $\epsilon = \epsilon(\theta)$ such that for all $x, y \in \mathcal{C}_K\mathcal{Y}$ we have

$$\mathsf{d}_Z(\mathfrak{p}_Z(x), \mathfrak{p}_Z(y)) - \epsilon \;\leqslant\; \mathsf{d}^{\mathfrak{p}}_Z(x, y) \;\leqslant\; \mathsf{d}_Z(\mathfrak{p}_Z(x), \mathfrak{p}_Z(y)). \tag{1}$$

We can take $\Theta(\theta) \geqslant 2\theta + \epsilon$.

A final point about the quasitree of metric spaces is that, similarly to HHSs, it is possible to approximate the distance between two points by summing the distances in the component metric spaces.

3.24 Theorem ([BBF15, Thm 4.13]). *There is a constant $K' = K'(K, \theta) > K$ such that*

$$\frac{1}{2}\sum_{Z\in\mathcal{Y}} \left\{\!\!\left\{\mathsf{d}^{\mathfrak{p}}_Z(x, y)\right\}\!\!\right\}_{K'} \;\leqslant\; \mathsf{d}_{\mathcal{C}_K\mathcal{Y}}(x, y) \;\leqslant\; 6K + 4\sum_{Z\in\mathcal{Y}} \left\{\!\!\left\{\mathsf{d}^{\mathfrak{p}}_Z(x, y)\right\}\!\!\right\}_{K}$$

for every $x, y \in \mathcal{C}_K\mathcal{Y}$.

One noteworthy point on Theorem 3.24 is that there is no flexibility in the values of K and K', unlike in the distance formula for HHSs. This will not cause problems for us.

3.3 Embedding colourable HHGs in products of hyperbolic spaces

We now aim to use the Bestvina–Bromberg–Fujiwara construction to embed colourable HHGs in products of hyperbolic spaces. The following lemma shows how to apply the Bestvina–Bromberg–Fujiwara construction to construct the hyperbolic spaces. Recall that s_0 is the minimal threshold constant from the distance formula.

3.25 Proposition. *If $(X, \mathfrak{S})$ is an HHS and $\mathfrak{S}' \subset \mathfrak{S}$ is a set of pairwise transverse domains, then the set $\{\mathcal{C}U : U \in \mathfrak{S}'\}$ with specified points $\{\rho^V_U : U, V \in \mathfrak{S}'\}$ satisfies the projection axioms with constant $s_0 + 2E$. Moreover, if G acts on X by HHS automorphisms and $\mathfrak{S}'$ is G–invariant, then the projection axioms are satisfied G–equivariantly.*

Proof. For Axiom (P1), suppose that $\mathsf{d}_W(\rho^U_W, \rho^V_W) > 2E$. Let x lie in the product region P_U. We have $\mathsf{d}_W(x, \rho^V_W) \geqslant \mathsf{d}_W(\rho^U_W, \rho^V_W) - E > E$, so the consistency inequality states that $\mathsf{d}_V(x, \rho^W_V) \leqslant E$. This shows that $\mathsf{d}_V(\rho^U_V, \rho^W_V) \leqslant \mathsf{d}_V(\rho^U_V, x) + \mathsf{d}_V(x, \rho^W_V) \leqslant 2E$.

Now, given $U, V \in \mathfrak{S}'$, let $x \in P_U$ and let $y \in P_V$. If $\mathsf{d}_W(\rho^U_W, \rho^V_W) \geqslant s_0 + 2E$, then $\mathsf{d}_W(x, y) \geqslant s_0$. By the distance formula, this can only hold for finitely many W, establishing Axiom (P2). The equivariance statement is obvious. □

3.26. In view of Proposition 3.25, if $(G, \mathfrak{S})$ is an HHG with a colouring $\mathfrak{S} = \bigsqcup_{i=1}^{\chi} \mathfrak{S}_i$, then we can build a quasitree of metric spaces $\mathcal{C}_K\mathfrak{S}_i$ for each i. According to Theorems 3.16 and 3.22, the $\mathcal{C}_K\mathfrak{S}_i$ are hyperbolic spaces with finite asymptotic dimension.

The goal of this section is to prove the following, which sums up Propositions 3.32 and 3.38.

3.27 Theorem. *Let $(G, \mathfrak{S})$ be an HHG with a colouring $\mathfrak{S} = \bigsqcup_{i=1}^{\chi} \mathfrak{S}_i$. There is a constant K_0 such that, for any $K \geqslant K_0$, there is a quasimedian, quasiisometric embedding $\Psi : G \to \prod_{i=1}^{\chi} \mathcal{C}_K\mathfrak{S}_i$.*

3.28 Remark. If the conclusion of Theorem 3.27 holds for G^{chr} (see Definition 3.17), then it follows for G because G^{chr} is coarsely dense in G. Since finite-index subgroups of HHGs are HHGs (with different constants), we shall therefore assume that the action of G on $\{\mathfrak{S}_i : 1 \leqslant i \leqslant \chi\}$ is trivial in the proof of Theorem 3.27.

With this assumption, the map Ψ that we shall produce in our proof of Theorem 3.27 will actually be *equivariant*. This equivariance is lost in passing to finite-index overgroups.

3.29 Assumption. For the remainder of this section, we assume that $(G,\mathfrak{S})$ is a colourable HHG, with $\mathfrak{S}=\bigsqcup_{i=1}^{\chi}\mathfrak{S}_i$ and $G^{\mathrm{chr}}=G$. That is, $g\mathfrak{S}_i=\mathfrak{S}_i$ for all i and all $g\in G$.

3.30 Construction (The map Ψ)**.**

Let D_1 be the constant from Lemma 3.8. Set $D=\max\{D_1,\Theta(s_0+2E)\}$ and $K_0=101D$. Fix a choice of $K\geqslant K_0$, and consider the quasitrees of metric spaces $\mathcal{C}\mathfrak{S}_i=\mathcal{C}_K\mathfrak{S}_i$.

It will be convenient to have made a certain choice of representative domain U_i for each colour i. Let $U\in\mathfrak{S}_i$, and let h lie in the product region P_U. Set $U_i=h^{-1}U$. We have that $1\in P_{U_i}$.

Now, for each i, define $\psi_i:G\to\mathcal{C}\mathfrak{S}_i$ by setting $\psi_i(g)=g\pi_{U_i}(1)$, and let $\Psi:G\to\prod_{i=1}^{\chi}\mathcal{C}\mathfrak{S}_i$ be given by $\Psi(g)=(\psi_1(g),\dots,\psi_\chi(g))$. That is, Ψ is simply the diagonal orbit map with respect to the basepoint $(\pi_{U_1}(1),\dots,\pi_{U_\chi}(1))$.

3.31 Lemma. *For any $g\in G$ and $U\in\mathfrak{S}_i$, we have $\mathsf{d}_U(g,\mathfrak{p}_U\psi_i(g))\leqslant E$.*

Proof. If $U=gU_i$, then $\mathfrak{p}_U\psi_i(g)=\pi_U(g)$. Otherwise, $U\pitchfork gU_i$, so $\mathfrak{p}_U\psi_i(g)=\rho_U^{gU_i}$. The fact that $1\in P_{U_i}$ implies that $g\in P_{gU_i}$, so the definition of a product region shows that $\mathsf{d}_U(g,\rho_U^{gU_i})\leqslant E$. □

Let us now prove that Ψ is a quasiisometric embedding. The argument follows [BBF15], where it is proved for mapping class groups.

3.32 Proposition. *Ψ is a quasiisometric embedding.*

Proof. Because Ψ is equivariant and G is finitely generated, we see from the triangle inequality that Ψ is coarsely Lipschitz. Let $s=\max\{s_0,K'+\epsilon+2E\}$, where ϵ is from inequality (1), s_0 is from the distance formula, and $K'>\epsilon+2E$ is from Theorem 3.24. Let $g\in G$. For any $U\in\mathfrak{S}_i$, Lemma 3.31 shows that $\mathsf{d}_U(1,g)\leqslant\mathsf{d}_U(\mathfrak{p}_U\psi_i(1),\mathfrak{p}_U\psi_i(g))+2E\leqslant\mathsf{d}_U^{\mathfrak{p}}(\psi_i(1),\psi_i(g))+\epsilon+2E$. In particular, if $\mathsf{d}_U(1,g)\geqslant s$, then $\mathsf{d}_U^{\mathfrak{p}}(\psi_i(1),\psi_i(g))\geqslant K'$. It follows that

$$2\left\{\!\!\left\{\mathsf{d}_U^{\mathfrak{p}}(\psi_i(1),\psi_i(g))\right\}\!\!\right\}_{K'}\ \geqslant\ \left\{\!\!\left\{\mathsf{d}_U^{\mathfrak{p}}(\psi_i(1),\psi_i(g))+\epsilon+2E\right\}\!\!\right\}_{K'}\ \geqslant\ \left\{\!\!\left\{\mathsf{d}_U(1,g)\right\}\!\!\right\}_s.$$

According to Theorem 3.24, this shows that $\mathsf{d}_{\mathcal{C}\mathfrak{S}_i}(\psi_i(1),\psi_i(g))$ is coarse-linearly lower bounded by $\sum_{U\in\mathfrak{S}_i}\{\!\{\mathsf{d}_U(1,g)\}\!\}_s$. The proof is completed by summing over $i\in\{1,\dots,\chi\}$ and applying the distance formula. □

It remains to show that Ψ is quasimedian. We use the following proposition.

3.33 Proposition. *There is a constant λ such that $\psi_i\gamma$ is an unparametrised λ–quasigeodesic for any D–hierarchy path $\gamma\subset G$.*

Let us fix some notation for the proof of Proposition 3.33. Fix a D–hierarchy path $\gamma:\{0,\dots,T\}\to G$. By equivariance, we may assume that $\gamma(0)=1$. Write $\gamma(T)=g$. For any $U\in\mathfrak{S}$, the path $\pi_U\gamma$ is an unparametrised D–quasigeodesic, so if $U\in\mathfrak{S}_i\cap\mathrm{Rel}_{100D}(1,g)$, then there exist a minimal $a_U\in\{0,\dots,T\}$ and a maximal $b_U\in\{0,\dots,T\}$ with $\mathsf{d}_U(\gamma(a_U),1),\mathsf{d}_U(\gamma(b_U),g)>2D$. Moreover, $\pi_U\gamma|_{[0,a_U]}$ and $\pi_U\gamma|_{[b_U,T]}$ are each $10D$–coarsely constant. Write $\gamma_U=\gamma|_{[a_U,b_U]}$. By consistency, $b_U<a_V$ whenever $U<V$ in the ordering of $\mathrm{Rel}_{100D}(1,g)$. Let us write $x_U=\gamma(a_U)$ and $y_U=\gamma(b_U)$.

3.34 Lemma. *If $U\in\mathfrak{S}_i\cap\mathrm{Rel}_{100D}(1,g)$, then $\mathsf{d}_{\mathcal{C}\mathfrak{S}_i}(\psi_i(x),\pi_U(x))\leqslant 6K$ for all $x\in\gamma_U$. In particular, there is a constant $D'=D'(D,K)$ such that $\psi_i\gamma_U$ is an unparametrised D'–quasigeodesic.*

Proof. If the conclusion does not hold, then Theorem 3.24 provides a domain $W \in \mathfrak{S}_i$ such that $\mathsf{d}^{\mathfrak{p}}_W(\psi_i(x), \pi_U(x)) \geqslant K$. According to inequality (1), it follows that $\mathsf{d}_W(\mathfrak{p}_W\psi_i(x), \mathfrak{p}_W\pi_U(x)) \geqslant K - \epsilon$. Lemma 3.31 precludes $U = W$, so $W \pitchfork U$.

If $W = xU_i$, then $\mathsf{d}_W(\mathfrak{p}_W\psi_i(x), \mathfrak{p}_W\pi_U(x)) = \mathsf{d}_W(x, \rho^U_W)$. Otherwise, (we are in the case illustrated in Figure 2, and)

$$\mathsf{d}_W(\mathfrak{p}_W\psi_i(x), \mathfrak{p}_W\pi_U(x)) = \mathsf{d}_W(\rho^{xU_i}_W, \rho^U_W) \leqslant \mathsf{d}_W(x, \rho^U_W) + E,$$

because $x \in P_{xU_i}$. In either case, consistency tells us that $\mathsf{d}_U(x, \rho^W_U) \leqslant E$. By definition of γ_U, this implies that $\pi_U(1)$ and $\pi_U(g)$ are both at distance greater than E from ρ^W_U. By consistency, $\pi_W(1)$ and $\pi_W(g)$ are both E–close to ρ^U_W, but this contradicts the fact that $\pi_W\gamma$ is an unparametrised quasigeodesic, because $\mathsf{d}_W(x, \rho^U_W) \geqslant K - \epsilon - E$. $\square$

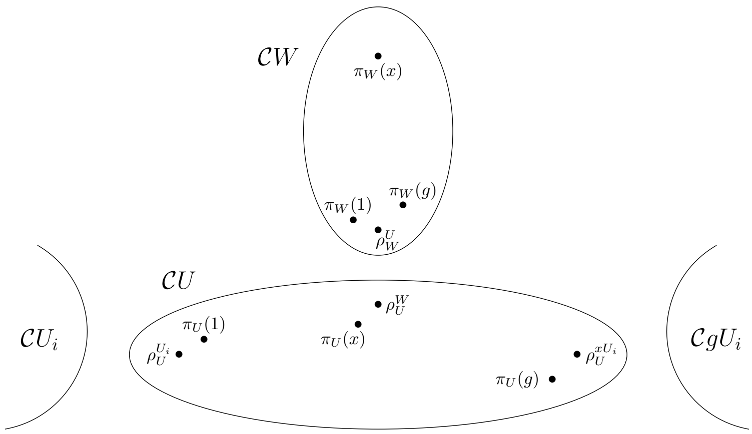

Figure 2: The proof of Lemma 3.34.

Now let $M > K$ be a large constant, to be specified later. Enumerate $\mathrm{Rel}_M(1, g) \cap \mathfrak{S}_i = \{U_2, U_4, \dots, U_n\}$ according to the total order. For simplicity, abbreviate a_{U_j} to a_j and y_{U_j} to y_j, etc. Let $b_0 = 0$, $a_{n+2} = T$, and for odd j set $\alpha_j = \gamma|_{[b_{j-1}, a_{j+1}]}$.

3.35 Lemma. *Each $\psi_i\alpha_j$ is an unparametrised quasigeodesic with constant independent of M.*

Proof. Let $\hat\alpha_j$ be a geodesic in $\mathcal{C}\mathfrak{S}_i$ from $\psi_i(y_{j-1})$ to $\psi_i(x_{j+1})$. It suffices to show that $\psi_i\alpha_j$ fellow-travels $\hat\alpha_j$ with constant independent of M.

Our choice of D (large in terms of E and Θ) allows us to invoke [BBF15, Thm 4.11] to get a constant $\lambda_1 = \lambda_1(E, K)$ such that if $U \in \mathrm{Rel}_{100D}(1, g) \cap \mathfrak{S}_i$ and $U_{j-1} < U < U_{j+1}$, then $\hat\alpha_j$ comes λ_1–close to $\mathfrak{p}_U\psi_i(y_{j-1})$ and $\mathfrak{p}_U\psi_i(x_{j+1})$.

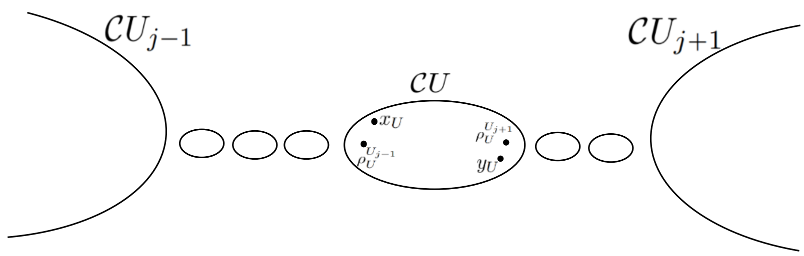

Figure 3: The proof of Lemma 3.35.

By [BBF15, Cor. 4.10], the map $\mathfrak{p}_U$ is coarsely Lipschitz, with constants depending only on E and K. Since $\mathfrak{p}_U\pi_{U_{j-1}}(y_{j-1}) = \rho_U^{U_{j-1}}$, this bounds $\mathsf{d}_{\mathcal{C}\mathfrak{S}_i}(\mathfrak{p}_U\psi_i(y_{j-1}), \rho_U^{U_{j-1}})$ in terms of $\mathsf{d}_{\mathcal{C}\mathfrak{S}_i}(\psi_i(y_{j-1}), \pi_{U_{j-1}}(y_{j-1}))$, which in turn is bounded by Lemma 3.34. But, by Lemma 3.34 and the definition of x_U, the set $\rho_U^{U_{j-1}}$ is uniformly close to $\psi_i(x_U)$ in terms of E and K. Thus $\mathfrak{p}_U\psi_i(y_{j-1})$ lies at uniformly bounded distance from $\psi_i(x_U)$, and similarly $\mathfrak{p}_U\psi_i(x_{j+1})$ lies at uniform distance from $\psi_i(y_U)$.

We have shown that $\hat{\alpha}_j$ comes uniformly close to both $\psi_i(x_U)$ and $\psi_i(y_U)$. According to Lemma 3.34, it follows that the path $\psi_i\alpha_j|_{[a_U,b_U]}$ fellow-travels a subgeodesic of $\hat{\alpha}_j$ with constant depending only on E and K. By the definition of total ordering on $\mathrm{Rel}_{100D}(1,g) \cap \mathfrak{S}_i$, there is a bound on the overlap between any two of these subpaths.

To complete the proof, it suffices to show that their complement is bounded. But if x and y are points on γ with $\mathfrak{S}_i \cap \mathrm{Rel}_{100D}(x,y) = \varnothing$, then Theorem 3.24 and Lemma 3.31 tell us that $\mathsf{d}_{\mathcal{C}\mathfrak{S}_i}(\psi_i(x), \psi_i(y))$ is bounded by $6K$, because $K \geqslant 101D$. $\square$

Our strategy for proving Proposition 3.33 will be to apply the following lemma for recognising when a piecewise-quasigeodesic in a hyperbolic space is a quasigeodesic, which extends a standard argument found in [HW15].

3.36 Lemma ([HW16, Lem. 4.3]). *For any constants δ, λ and any function f, there is a constant L_0 satisfying the following. Suppose that P is a Lipschitz path in a δ–hyperbolic space that decomposes as a concatenation $P = \alpha_1\gamma_2\alpha_3 \dots \gamma_n\alpha_{n+1}$ of λ–quasigeodesics. Suppose further that, for all $r > 3\delta$, the sets*

$$\mathcal{N}(\gamma_j, r) \cap \gamma_{j\pm2} \text{ and } \mathcal{N}(\gamma_k, r) \cap \alpha_{k\pm1}$$

have diameter at most $f(r)$. If every γ_j has length at least L_0, then P is an L_0–quasigeodesic.

3.37 Lemma. *There is a function f, independent of M, such that the subsets*

$$\mathcal{N}_{\mathcal{C}\mathfrak{S}_i}(\psi_i\gamma_j, r) \cap \psi_i\gamma_{j\pm2} \quad \text{and} \quad \alpha_k \cap \mathcal{N}_{\mathcal{C}\mathfrak{S}_i}(\psi_i\gamma_{k\pm1}, r)$$

of $\mathcal{C}\mathfrak{S}_i$ have diameters bounded by $f(r)$.

Proof. According to [BBF15, Thm A], if $U \in \mathfrak{S}_i$, then $\mathcal{C}U$ is quasiconvex in $\mathcal{C}\mathfrak{S}_i$, so closest-point projection to $\mathcal{C}U$ is coarsely Lipschitz. Furthermore, [BBF15, Cor. 4.10] tells us that the closest-point projection of $\mathcal{C}U_j$ onto $\mathcal{C}U_{j\pm2}$ in $\mathcal{C}\mathfrak{S}_i$ is contained in a uniform neighbourhood of $\rho_{U_{j\pm2}}^{U_j}$. It follows that $\mathcal{N}_{\mathcal{C}\mathfrak{S}_i}(\mathcal{C}U_j, r) \cap \mathcal{C}V$ has diameter bounded in terms of r. This is enough for the first intersection by Lemma 3.34.

Again because of Lemma 3.34, this moreover shows that the closest point in $\mathcal{C}U_{k+1}$ to $\psi_i(y_{k-1})$ is uniformly close to $\rho_{U_{k+1}}^{U_{k-1}}$. From the definition of x_{k+1} and Lemma 3.34, we also know that $\psi_i(x_{k+1})$ is uniformly close to $\rho_{U_{k+1}}^{U_{k-1}}$. Thus the closest point projection of $\psi_i\alpha_k$ to $\mathcal{C}U_{k+1}$ is a coarse point, as $\psi_i\alpha_k$ is a quasigeodesic from $\psi_i(y_{k-1})$ to $\psi_i(x_{k+1})$ (Lemma 3.35), and similarly for $\mathcal{C}U_{k-1}$. This gives the bound for the second intersection. $\square$

We can now prove Proposition 3.33.

Proof of Proposition 3.33. γ is a D–quasigeodesic, so by Proposition 3.32 and the fact that coordinate maps are Lipschitz, $\psi_i\gamma$ is a Lipschitz path.

In light of Lemmas 3.34, 3.35, and 3.37, the conditions of Lemma 3.36 are met by $\psi_i\gamma = (\psi_i\alpha_1)(\psi_i\gamma_2)\dots(\psi_i\alpha_{n+1})$, with all parameters in terms of E and K only. There thus exists a constant $L_0 = L_0(E,K)$ such that if every $\psi_i\gamma_j$ has length at least L_0, then $\psi_i\gamma$ is an L_0–quasigeodesic.

Now fix $M = L_0 + 20K$. The length of $\psi_i\gamma_j$ is at least the distance between its endpoints, $\psi_i(x_j)$ and $\psi_i(y_j)$. By construction and Lemma 3.34, these are, respectively, $10D + 6K$–close to $\pi_{U_j}(1)$ and $\pi_{U_j}(g)$. We have bounded below the length of $\psi_i\gamma_j$ by $\mathsf{d}_U(1,g) - 20D - 12K \geqslant L_0 + 20K - 10D - 12K > L_0$, which completes the proof. $\square$

3.38 Proposition. Ψ *is quasimedian.*

Proof. It is enough to prove that the ψ_i are quasimedian. Given $x_1, x_2, x_3 \in G$, let $m = \mu(x_1, x_2, x_3)$. As $D \geqslant D_1$, Lemma 3.8 shows that there are D–hierarchy paths γ_{jk} from x_j to x_k that pass through m. Proposition 3.33 shows that the $\psi_i\gamma_{jk}$ form a uniform quasigeodesic triangle all of whose sides contain $\psi_i(m)$. Since $\mathcal{C}\mathfrak{S}_i$ is hyperbolic, the Morse lemma tells us that $\psi_i(m)$ is uniformly close to $\mu_{\mathcal{C}\mathfrak{S}_i}(\psi_i(x_1), \psi_i(x_2), \psi_i(x_3))$. $\square$

4 Quasicubicality

This section is based on [Pet21]. The end goal is to prove that colourable HHGs admit quasimedian quasiisometries to finite-dimensional CAT(0) cube complexes, but the only way hierarchical hyperbolicity is used directly is by invoking Theorem 3.27. In order to make use of that result, we characterise when hyperbolic spaces admit such quasiisometries. The proofs of both results conclude by applying Proposition 2.13.

In Section 4.1, we describe a construction due to Buyalo–Dranishnikov–Schroeder for quasiisometrically embedding certain *hyperbolic cones* in finite products of trees [BDS07]. This is part of a broader circle of ideas [Dra03, Buy05b, LS05, BS00, BP03]. The purpose of Section 4.2 is to show that this embedding is quasimedian. In a sense this is just an observation, because Buyalo–Dranishnikov–Schroeder certainly had the necessary tools at their disposal, but they were writing before the importance of medians had become apparent in a coarse geometry context. In Section 4.3, we apply the construction to hyperbolic spaces and colourable HHGs. We conclude with Section 4.4, which discusses the special case of mapping class groups and some properties of the cube complexes they are quasiisometric to.

4.1 Embedding hyperbolic cones in finite products of trees

Here we describe the construction of [BDS07, §7–9]. Given a complete, bounded metric space Z, the first step is to construct a hyperbolic graph $\operatorname{Co} Z$, called the *hyperbolic cone on Z*. The second step is to show that if Z has finite *capacity dimension*, then $\operatorname{Co} Z$ can be quasiisometrically embedded in a finite product of trees. Capacity dimension was introduced by Buyalo in [Buy05a]; the notion itself will not be important for us, as we shall replace it by asymptotic dimension when we apply the construction, so the reader is referred to [Buy05a] for a definition.

4.1 Definition (Separated, net)**.** Let X be a metric space and let $r > 0$. A subset Y is said to be *r–separated* if $\mathsf{d}(y, y') \geqslant r$ for all distinct $y, y' \in Y$, and Y is an *r–net* if it is r–coarsely dense.

Let Z be a complete, bounded metric space. After rescaling Z, we shall assume that $\operatorname{diam} Z < 1$.

4.2 Hyperbolic cones. Fix a positive constant $r \leqslant \frac{1}{6}$, and for each integer $k \geqslant 0$ let V_k be an r^k–separated r^k–net in Z. The hyperbolic cone is not canonical, in the sense that it depends on these choices. In particular, the properties of the cone will depend on the value of r, which we call the *parameter* of the cone. On the other hand, the choices of net are immaterial, so we shall have no qualms about calling it “the” hyperbolic cone on Z (with parameter r).

Associate with each $v \in V_k$ the ball $B(v) = B_Z(v, 2r^k)$. Observe that since $\operatorname{diam} Z < 1$, the separated net V_0 is a singleton. Let us write $V_0 = \{o\}$. We have $B(o) = Z$.

The hyperbolic cone $\operatorname{Co} Z$ on Z is a graph. Its vertex set is $V = \bigsqcup_{k \geqslant 0} V_k$. Vertices $v_1 \in V_{k_1}$ and $v_2 \in V_{k_2}$ are joined by an edge if either of the following holds.

- $k_1 = k_2$ and the closed balls $\overline{B}(v_1)$ and $\overline{B}(v_2)$ intersect.
- $k_1 = k_2 - 1$ and $B(v_2) \subset B(v_1)$.

The *level* $\ell(v)$ of $v \in V$ is the unique k such that $v \in V_k$.

The next fact justifies the name “hyperbolic cone”.

4.3 Proposition ([BP03, Prop. 2.1],[BDS07, Thm 7.4]). *If Z is a complete and bounded metric space, then* $\operatorname{Co} Z$ *is hyperbolic, and the hyperbolicity constant depends only on* $\operatorname{diam} Z$ *and the parameter r.*

Following [BDS07, §8,9], we now describe how to construct the product of trees. The process starts by building a sequence of coloured open covers of our bounded metric space Z. Each colour gives rise to a tree.

4.4 Definition (Disjoint, colouring). A collection of subsets of Z is said to be *disjoint* if no two of its members intersect. An *n–colouring* of a collection $\mathcal{V}$ of subsets of Z is a finite decomposition $\mathcal{V} = \bigcup_{c \in C} \mathcal{V}^c$, with $|C| = n$, such that each $\mathcal{V}^c$ is disjoint. *Cf.* Definition 2.4.

Note that an n–colouring need not be a partition. The trees will be built from the colours in the following proposition. More specifically, given a colour c, the vertices will be the elements of the sequence of covers that are coloured c, and we shall use the sequence to add edges.

4.5 Proposition ([Buy05b, Prop. 2.3], [BDS07, Thm 8.2]). *Let Z be a complete, bounded metric space with capacity dimension n. There is a constant $\varepsilon \in (0,1)$ such that for any sufficiently small $r \in (0, \frac{\varepsilon}{4})$ there exists a sequence $(\mathcal{U}_k = \bigcup_{c \in C} \mathcal{U}_k^c)_{k \geqslant 0}$ of $(n+1)$–coloured open covers of Z such that, for any hyperbolic cone on Z with parameter r, the following are satisfied.*

(C1) $\sup\{\operatorname{diam}_Z(U) : U \in \mathcal{U}_k\} < r^k$ *for every k. Moreover, $\mathcal{U}_0^c = \{Z\}$ for all $c \in C$.*

(C2) For every $v \in V_{k+1}$ there exists $U \in \mathcal{U}_k$ such that $B(v) \subset U$.

(C3) For every $c \in C$ and for any distinct $U \in \mathcal{U}_k^c$ and $U' \in \mathcal{U}_{k'}^c$ with $k \leqslant k'$, we have that $\mathcal{N}_Z(U', \varepsilon r^{k'})$ is either disjoint from U or is a subset of it.

Our arguments will not make explicit use of (C2), though it is used by Buyalo–Dranishnikov–Schroeder to prove Proposition 4.9.

4.6. Define $\mathcal{U}^c = \bigsqcup_{k \geqslant 0} (\mathcal{U}_k^c \times \{k\})$. Formally, an element of $\mathcal{U}^c$ is a pair (U, k), where $U \in \mathcal{U}_k^c$, but we shall often abuse notation slightly by just writing $U \in \mathcal{U}^c$. We call k the *level* of U, and denote it by $\ell(U)$, just as with elements of V. Let $\mathcal{U} = \bigsqcup_{c \in C} \mathcal{U}^c$. See Figure 4.

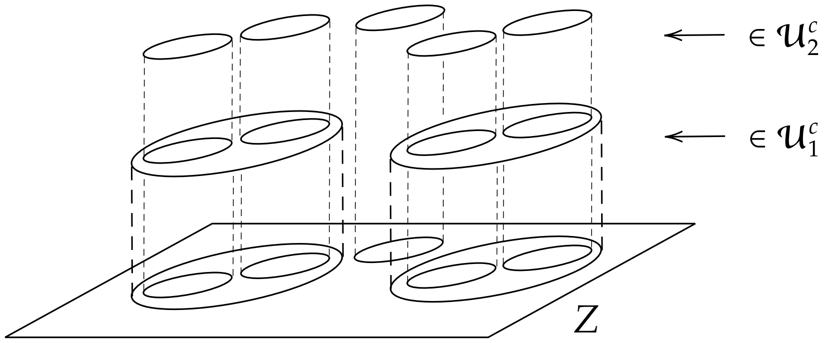

Figure 4: Schematic picture of one of the $\mathcal{U}^c$.

Now, in the sense of Proposition 4.5, fix a sufficiently small constant $r < \min\{\frac{1}{7}, \frac{\varepsilon}{4}\}$, a sequence $(\mathcal{U}_k)_{k \geqslant 0}$ of coloured covers of Z, and a hyperbolic cone on Z with parameter r. To improve clarity, let us write $\mathcal{U}_0^c = \{o_c\}$. Of course, (C1) states that, as subspaces of Z, we have $o_c = Z$ for all c.

4.7 Construction (Trees)**.** For each colour c we build a rooted tree T_c. The vertex set of T_c is $\mathcal{U}^c$, and the root is o_c. We join vertices (U, k) and (U', k') with $k < k'$ by an edge if $U' \subset U$ as subsets of Z and there is no $(U'', k'') \in \mathcal{U}^c$ with $U' \subset U''$ and $k < k'' < k'$.

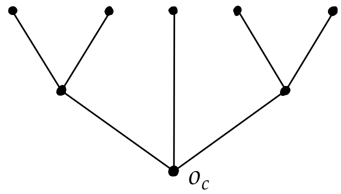

Figure 5: The part of T_c corresponding to Figure 4.

We are ready to define a map $f_c : \operatorname{Co} Z \to T_c$.

4.8 Construction (Maps)**.** Set $f_c(o) = o_c$. For $v \in V_k$ with $k > 0$, define $f_c(v) = U \in \mathcal{U}^c$ to be the element of maximal level that has $B(v) \subset U$. This exists because $o_c = Z$, and it is well defined by disjointness of each $\mathcal{U}^c_j$. Each point $x \in \operatorname{Co} Z \smallsetminus V$ lies on some edge vv'—choose $f_c(x) \in \{f_c(v), f_c(v')\}$ arbitrarily.

4.9 Proposition ([BDS07, Lem. 9.9, Thm 9.2])**.** *Suppose that Z is a complete, bounded metric space with capacity dimension n. The maps $f_c|_V : V \to T_c$ are 2–Lipschitz, and $(f_c)_{c \in C} : \operatorname{Co} Z \to \prod_{c \in C} T_c$ is a quasiisometric embedding of $\operatorname{Co} Z$ in a product of $n + 1$ trees.*

4.2 The embedding is quasimedian

We start with a couple of simple preliminary lemmas. With each $U \in \mathcal{U}_k$ we associate a subset $\hat{U} \subset V_k$ by setting $\hat{U} = \{v \in V_k : B(v) \cap U \neq \varnothing\}$. Recall that if s and t are vertices of a tree T, then $[s, t]$ is the unique geodesic between them.

4.10 Lemma. *Suppose that $U \in \mathcal{U}^c_k$ and that $v \in \hat{U}$. We have $f_c(v) \in [o_c, U] \cap B_{T_c}(U, 2)$.*

Proof. As noted in [BDS07, §9.3], the fact that $B(v)$ has radius $2r^k$, which is greater than $\sup\{\operatorname{diam}_Z(W) : W \in \mathcal{U}_j\}$ for all $j \geqslant k$, implies that $\ell(f_c(v)) < k$. Since $B(v)$ intersects U, property (C3) of $\mathcal{U}$ tells us that $f_c(v) \in [o_c, U]$.

If $\mathsf{d}_{T_c}(o_c, U) \leqslant 2$ then we are done. Otherwise, let U', U'' be the vertices of $[o_c, U]$ with $\mathsf{d}_{T_c}(U, U') = 1$ and $\mathsf{d}_{T_c}(U, U'') = 2$. We have a chain of subsets of Z as follows:

$$\begin{aligned} B(v) \subset \mathcal{N}_Z(U, 4r^k) \subset \mathcal{N}_Z(U', 4r^k) \subset \\ \subset \mathcal{N}_Z(U', \varepsilon r^{k-1}) \subset \mathcal{N}_Z(U', \varepsilon r^{\ell(U')}) \subset U''. \end{aligned}$$

By definition of f_c, we have that $f_c(v) \in \{U', U''\}$. □

4.11 Lemma. *For every $U \in \mathcal{U}$, the set $\hat{U}$ is nonempty and has $\operatorname{diam}_{\operatorname{Co} Z}(\hat{U}) \leqslant 2$.*

Proof. Nonemptiness is automatic because V_k is an r^k–net. Let $v \in \hat{U}$. There exists $v^- \in V_{k-1}$ such that $\mathsf{d}_Z(v^-, v) \leqslant r^{k-1}$. Now, if $z \in B(v')$ for some $v' \in \hat{U}$, then

$$\begin{aligned} \mathsf{d}_Z(z, v^-) &\leqslant \mathsf{d}_Z(z, U) + \operatorname{diam}_Z(U) + \mathsf{d}_Z(U, v) + \mathsf{d}_Z(v, v^-) \\ &\leqslant 4r^k + r^k + 2r^k + r^{k-1}, \end{aligned}$$

which is less than $2r^{k-1}$ because $r < \frac{1}{7}$. Thus $B(v') \subset B(v^-)$ for all $v' \in \hat{U}$, so every $v' \in \hat{U}$ is joined to v^- by an edge of $\operatorname{Co} Z$. Thus $\operatorname{diam}_{\operatorname{Co} Z}(\hat{U}) \leqslant 2$. □

We can now establish that the map of Proposition 4.9 is quasimedian (Definition 2.12).

4.12 Proposition. *Suppose that Z is a complete, bounded metric space with capacity dimension n. The map $(f_c)_{c\in C} : \operatorname{Co} Z \to \prod_{c\in C} T_c$ of Construction 4.8 is quasimedian.*

Proof. We must show that each factor map f_c is quasimedian, and it is enough to work with the restriction $f_c|_V$ to the coarsely dense subset $V \subset \operatorname{Co} Z$. According to Lemma 2.17, it suffices to show that f_c is a coarse morphism for the binary operations $\mu_{\operatorname{Co} Z}(\cdot,\cdot,o)$ and $\mu_{T_c}(\cdot,\cdot,o_c)$. Let x_1 and x_2 be vertices of $\operatorname{Co} Z$, and write $k_i = \ell(x_i)$.

For any $v \in V_k$ and any $j < k$, the fact that V_j is an r^j–net implies that there is some $v_j \in V_j$ with $\mathsf{d}_Z(v, v_j) \leqslant r^j$. For any $z \in B(v)$, we have

$$\mathsf{d}_Z(z, v_j) \leqslant \mathsf{d}_Z(z, v) + \mathsf{d}_Z(v, v_j) \leqslant 2r^k + r^j < 2r^j.$$

Hence $B(v) \subset B(v_j)$. Applying this to x_1 and x_2 shows that we can fix geodesics $\gamma_i = (o = x_{i_0}, x_{i_1}, \dots, x_{i_{k_i}} = x_i)$, from o to x_i inside $\operatorname{Co} Z$, such that $\ell(x_{i_j}) = j$. We have $B(x_{i_j}) \subset B(x_{i_{j-1}})$ for all i, j.

Recall that for a vertex $x \in V$, the image $f_c(x)$ is defined to be the element $U \in \mathcal{U}^c$ of maximal level such that $B(x) \subset U$. Let us write $U_i = f_c(x_i)$ and $U_{i_j} = f_c(x_{i_j})$. If $j_1 \leqslant j_2$, then $B(x_{i_{j_2}}) \subset B(x_{i_{j_1}})$, so $U_{i_{j_2}} \subset U_{i_{j_1}}$, by property (C3) of $\mathcal{U}$. Hence $f_c\gamma_i$ is a monotone map to the unique geodesic $[o_c, U_i]$ in T_c.

The median $\mu_{T_c}(U_1, U_2, o_c)$ is the maximal-level element $U_{12} \in [o_c, U_1] \cap [o_c, U_2]$. In other words, it is the element $U_{12} \in \mathcal{U}^c$ of maximal level such that $B(x_1) \cup B(x_2) \subset U_{12}$. Let us write $k_{12} = \ell(U_{12})$.

Let δ be a hyperbolicity constant for $\operatorname{Co} Z$, which exists by Proposition 4.3. Fix a geodesic γ_{12} in $\operatorname{Co} Z$ from x_1 to x_2, and define

$$M = \big\{x \in \operatorname{Co} Z : \max\{\mathsf{d}_{\operatorname{Co} Z}(x, \gamma_1), \mathsf{d}_{\operatorname{Co} Z}(x, \gamma_2), \mathsf{d}_{\operatorname{Co} Z}(x, \gamma_{12})\} \leqslant 2\delta + 2\big\}.$$

Because $\operatorname{Co} Z$ is δ–hyperbolic, it is $2\delta + 2$–hyperbolic. Thus $\mu_{\operatorname{Co} Z}(x_1, x_2, o) \in M$, and the diameter of M is bounded by some constant $D = D(\delta)$. Note that $M \cap \gamma_i \neq \varnothing$.

Because U_{12} contains $B(x_i)$, it intersects $B(x_{i_{k_{12}}})$. Thus $x_{i_{k_{12}}} \in \hat{U}_{12}$, so Lemma 4.10 tells us that $f_c(x_{i_{k_{12}}}) \in [o_c, U_{12}] \cap B_{T_c}(U_{12}, 2)$. See Figure 6.

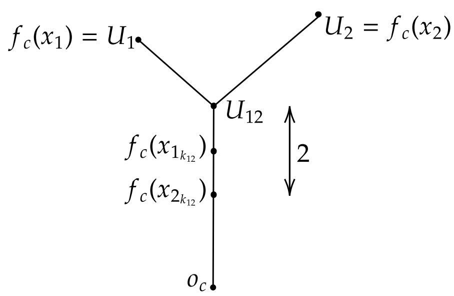

Figure 6: The images in T_c of various points of $\operatorname{Co} Z$.

Moreover, from Lemma 4.11, we have that $\mathsf{d}_{\operatorname{Co} Z}(x_{1_{k_{12}}}, x_{2_{k_{12}}}) \leqslant 2$. In particular, $x_{i_{k_{12}}}$ is 2–close to both γ_1 and γ_2. Now let y_i be the unique element of $M \cap \gamma_i$ of maximal level, which has $\ell(y_i) \geqslant \ell(x_{i_{k_{12}}}) = k_{12}$. Since $f_c\gamma_i$ is monotone in $[o_c, U_i]$, we have that $f_c(y_i) \in [f_c(x_{i_{k_{12}}}), U_i]$, and it follows that

$$\mathsf{d}_{T_c}\big(\mu_{T_c}(f_c y_1, f_c y_2, o_c),\ U_{12}\big) \leqslant 2.$$

Because y_1, y_2, and $m = \mu_{\mathrm{Co}\,Z}(x_1, x_2, o)$ all lie in M, the fact that $f_c|_V$ is 2–Lipschitz (Proposition 4.9) shows that $\mathsf{d}_{T_c}(f_c(y_i), f_c(m)) \leqslant 2D$. To sum up, we have

$$\begin{aligned}
&\mathsf{d}_{T_c}\left(\mu_{T_c}(f_c x_1, f_c x_2, o_c),\ f_c(\mu_{\mathrm{Co}\,Z}(x_1, x_2, o))\right)\\
&\qquad = \ \mathsf{d}_{T_c}(U_{12}, f_c(m))\\
&\qquad \leqslant \ \mathsf{d}_{T_c}\left(\mu_{T_c}(f_c y_1, f_c y_2, o_c),\ f_c(m)\right) + 2\\
&\qquad \leqslant \ \mathsf{d}_{T_c}\left(\mu_{T_c}(f_c m, f_c m, o_c), f_c(m)\right) + 2 + 4D\\
&\qquad = \ 2 + 4D,
\end{aligned}$$

because μ_{T_c} is 1–Lipschitz in each coordinate and $\mu_{T_c}(a, a, b) = a$. □

4.3 Application to hyperbolic spaces and colourable HHGs

Here we apply Proposition 4.12 to hyperbolic spaces of finite asymptotic dimension. We then use this together with Theorem 3.27 to show that colourable hierarchically hyperbolic groups are quasimedian quasiisometric to finite-dimensional CAT(0) cube complexes. Let us introduce some terminology to simplify future discussion.

4.13 Definition (Quasicubical)**.** A coarse median space X is *quasicubical* if there is a finite-dimensional CAT(0) cube complex Q and a quasimedian quasiisometry $X \to Q$.

We first collect a few facts about hyperbolic spaces.

4.14 Definition (Visual)**.** A hyperbolic space X is *visual* if, for some $x_0 \in X$, every $x \in X$ lies on a geodesic ray emanating from x_0.

The above definition of visual is a strengthening of a definition due to Bonk–Schramm [BS00]: any hyperbolic space that is visual in this sense is clearly visual in the sense of [BS00], and in the sense of [BDS07].

The following connects hyperbolic spaces with hyperbolic cones. Recall that ∂X denotes the *Gromov boundary* of a hyperbolic space X [Gro87].

4.15 Proposition ([BDS07, Thm 7.1], [BS00, Prop. 5.6])**.** *If X is a visual hyperbolic space, then there is a quasiisometry $X \to \mathrm{Co}(\partial X)$.*

Because we care about hyperbolic spaces that might not be visual and only need quasiisometric embeddings, we can afford to modify our starting space to make it visual.

4.16 Lemma (Visualisation)**.** *Every hyperbolic space X admits a median-preserving isometric embedding in a visual hyperbolic space Y with* $\operatorname{asdim} Y = \max\{1, \operatorname{asdim} X\}$.

Proof. Given X, let Y be the hyperbolic space obtained by attaching a ray $r_x = [0, \infty)$ to each $x \in X$. Clearly the inclusion map $X \hookrightarrow Y$ is median-preserving and isometric. Fix $x_0 \in X \subset Y$. We see that Y is visual by concatenating a geodesic from x_0 to x with the geodesic ray r_x.

According to [BD08, Prop. 23], the asymptotic dimension of a subspace of Y is bounded above by the asymptotic dimension of Y. In particular, $\operatorname{asdim} Y \geqslant \max\{\operatorname{asdim} r_x, \operatorname{asdim} X\}$. The upper bound on $\operatorname{asdim} Y$ is given by [BD08, Thm 25]. □

The next proposition lets us replace capacity dimension by asymptotic dimension.

4.17 Proposition ([MS13, Prop. 3.6])**.** *If Y is a geodesic hyperbolic space, then the capacity dimension of ∂Y is at most* $\operatorname{asdim} Y$.

4.18 Corollary. *For any hyperbolic space X with finite asymptotic dimension, there is a complete, bounded metric space Z with finite capacity dimension and such that there is a quasiisometric embedding $X \to \operatorname{Co} Z$.*

Proof. Given X as in the statement, let Y be the visual hyperbolic space produced by Lemma 4.16, which has $\operatorname{asdim} Y \leqslant 1 + \operatorname{asdim} X$. Let $Z = \partial Y$. As the boundary of a hyperbolic space, Z is complete and bounded [BS00, Prop. 6.2], and Proposition 4.17 shows that the capacity dimension of Z is finite. The concatenation $X \hookrightarrow Y \to \operatorname{Co} Z$ is a quasiisometric embedding by Proposition 4.15. □

We can now prove our results on quasicubicality of hyperbolic spaces and colourable hierarchically hyperbolic groups.

4.19 Theorem. *If X is hyperbolic, then X is quasicubical if and only if* $\operatorname{asdim} X < \infty$.

Proof. The forward direction holds because asymptotic dimension is preserved by quasiisometries [BD08, Prop. 22] and Wright showed that the asymptotic dimension of a CAT(0) cube complex is bounded by its cubical dimension [Wri12].

For the reverse direction, Corollary 4.18 shows that X quasiisometrically embeds in a hyperbolic cone $\operatorname{Co} Z$, and this embedding is automatically quasimedian by Lemma 2.15. Propositions 4.9 and 4.12 show that $\operatorname{Co} Z$, in turn, admits a quasimedian quasiisometric embedding in a finite product of trees. According to Proposition 2.13, this implies that X is quasicubical. □

For hyperbolic *groups*, Haglund–Wise showed the stronger result that they are always quasiisometric to locally finite CAT(0) cube complexes [HW12, Thm 1.8]. Theorem 4.19 extends this, because all hyperbolic groups have finite asymptotic dimension [Gro93, p.31]. It is not possible to get local finiteness in the generality of Theorem 4.19, though, because it applies, for example, to a regular tree of infinite valence.

4.20 Theorem. *Colourable HHGs are quasicubical.*

Proof. Let $(G, \mathfrak{S})$ be a colourable HHG. According to Theorem 3.27, there is a quasimedian quasiisometric embedding $G \to \prod_{i=1}^{\chi} \mathcal{C}_K \mathfrak{S}_i$. By Theorems 3.16 and 3.22, the $\mathcal{C}_K \mathfrak{S}_i$ are hyperbolic spaces with finite asymptotic dimension. Applying Theorem 4.19, we obtain finite-dimensional CAT(0) cube complexes $Q_1, \dots, Q_\chi$ such that Q_i is quasiisometric to $\mathcal{C}_K \mathfrak{S}_i$, and these quasiisometries are automatically quasimedian by Lemma 2.15. We therefore have a quasimedian quasiisometric embedding

$$G \to \prod_{i=1}^{\chi} \mathcal{C}_K \mathfrak{S}_i \to \prod_{i=1}^{\chi} Q_i$$

of G in a finite-dimensional CAT(0) cube complex. Proposition 2.13 shows that G is quasicubical. □

4.4 Mapping class groups and unusual cube complexes

Perhaps the most interesting special case of Theorem 4.20 is that of mapping class groups. For one thing, it is well known that only the simplest mapping class groups can act properly by semisimple isometries on CAT(0) spaces [KL96, Thm 4.2], and they also cannot act properly on infinite-dimensional CAT(0) cube complexes [Gen19, Thm 1.9]. However, Theorem 4.20 shows that they do *quasiact* properly and coboundedly on finite-dimensional CAT(0) cube complexes.

4.21. There are a few previous results of a similar flavour for mapping class groups. Indeed, the idea to use hyperbolic cones to prove embedding results for mapping class groups comes from Hume [Hum17], who proved that mapping class groups can be quasiisometrically embedded in finite products of trees. This is a non-asymptotic version of a result of Behrstock–Druţu–Sapir [BDS11] (also see [Bow14a]). An equivariant but more coarse result was proved by Bestvina–Bromberg–Fujiwara [BBF21], who showed that mapping class groups admit isometric actions on finite products of quasitrees such that the orbit maps are quasiisometric embeddings. A proof of quasicubicality of mapping class groups based on this can be found in [Pet21]. Furthermore, Hamenstädt constructed uniformly locally finite CAT(0) cube complexes with proper, coarsely Lipschitz coarse surjections to mapping class groups [Ham21], though these maps are not known to be quasiisometries.

4.22. Another reason that the specialisation to mapping class groups is interesting is that they satisfy *strong quasiisometric rigidity* [BKMM12]. For quasiisometries f, g of a metric space X, write $f \sim g$ if the function $\mathsf{d}(f(\cdot), g(\cdot)) : X \to \mathbf{R}$ is bounded. The quasiisometry group of X is

$$\mathrm{QI}(X) = {}^{\{\text{quasiisometries of } X\}}\!/\!_{\sim}.$$

Behrstock–Kleiner–Minsky–Mosher proved that, except in a few small cases, $\mathrm{QI}(\mathrm{MCG}\, S) \cong \mathrm{MCG}\, S$, and hence the same holds for any metric space quasiisometric to $\mathrm{MCG}\, S$. This shows that if Q is a CAT(0) cube complex quasiisometric to $\mathrm{MCG}(S)$, such as the finite-dimensional examples provided by Theorem 4.20, then

- $\mathrm{QI}(Q)$ is an infinite discrete group;
- Q is quasiisometric to a finitely generated group, but no CAT(0) cube complex quasiisometric to Q admits a proper cobounded group action [KL96].

There are examples of CAT(0) cube complexes with either one of these properties, but I am not aware of other examples that have both.

I thank Jingyin Huang for generously sharing and explaining his knowledge of the below examples.

4.23. As in [NR97], cube complexes that are quasiisometric to groups but not to cocompactly cubulated groups are given by *uniform lattices* $\Gamma < \mathrm{Sp}(n, 1)$. These are hyperbolic, so are quasiisometric to CAT(0) cube complexes Q_Γ by [HW12, Thm 1.8] or Theorem 4.19. They have *property (T)* by work of Kazhdan (see [BdlHV08, §3.3]) and Kostant [Kos75]. By Pansu's rigidity theorem [Pan89], if Q_Γ admitted a proper cobounded group action, then Γ would act with unbounded orbits on some CAT(0) cube complex, contradicting a result of Niblo–Reeves for groups with property (T) [NR97, Thm B]. On the other hand, Schwartz's theorem [Sch95] implies that $\mathrm{QI}(\Gamma)$ is isomorphic to the *commensurator* of Γ. By Corlette [Cor92] or Gromov–Schoen [GS92], Γ is *arithmetic*, so Margulis' characterisation of arithmeticity [Mar75, Thm 9] (also see [Zim84, §6.2]) implies that the commensurator of Γ is Hausdorff-dense in $\mathrm{Sp}(n, 1)$. Hence Q_Γ has indiscrete quasiisometry group.

4.24. For an example with infinite discrete quasiisometry group, let $\Lambda < \mathrm{SO}(n, 1)$ be a nonarithmetic nonuniform lattice, which exists by [GPS88]. The group Λ is hyperbolic *relative to virtually abelian subgroups*, so by *residual finiteness*, Λ is virtually a colourable HHG [BHS19, Thm 9.1], and hence is quasiisometric to a CAT(0) cube complex Q_Λ by Theorem 4.20. By Margulis' characterisation, Λ has finite index in its commensurator, so the quasiisometry group of Q_Λ is discrete by Schwartz's theorem. Whether Λ can virtually act on a CAT(0) cube complex is unknown in general, but Wise showed that Λ is virtually *compact special*, hence cocompactly cubulated, when $n = 3$ [Wis21, Thm 17.14].

5 Median–quasiconvexity and packing

In this section, we prove a simple coarse Helly result for quasicubical coarse median spaces; it also appears in [Pet21]. The functional statement is Corollary 5.6. In Section 5.2, we use this and Theorem 4.20 to deduce packing properties of median-quasiconvex subgroups of colourable HHGs, just as in [HHP20].

Convex subsets are important throughout geometry. In coarse settings, such as that of hyperbolic spaces, it is natural to consider *quasiconvexity* instead; this can be thought of as a notion of convexity that allows for some bounded error. For coarse median spaces, the appropriate version is *median-quasiconvexity.*

5.1 Definition (Median-quasiconvexity)**.** A subset Y of a coarse median space X is k–*median-quasiconvex* if $\mathsf{d}(Y, \mu(y, y', x)) \leqslant k$ for all $y, y' \in Y$, $x \in X$.

For HHSs, median quasiconvexity can also be characterised in terms of the hierarchy structure [RST18, Prop. 5.11], and in that context it is often referred to as *hierarchical quasiconvexity* [BHS19, Def. 5.1]. Let us now list some examples.

5.2 Examples of median-quasiconvexity.

- A full subcomplex of a CAT(0) cube complex is convex if and only if it is 0–median-quasiconvex (a.k.a. median-convex).
- A subspace of a hyperbolic space is quasiconvex if and only if it is median-quasiconvex.
- Bounded subsets are median-quasiconvex.
- Morse quasigeodesics in HHSs are median-quasiconvex [RST18].
- If X is a coarse median space of finite rank, then every subset of X has a (coarsely unique) median-quasiconvex hull [Bow18, Prop. 6.2].
- Multicurve stabilisers are median-quasiconvex in mapping class groups [BHS19, Prop. 5.11].
- If M is a closed 3–manifold without Nil or Sol components, then the cut tori in the prime decomposition of M form median-quasiconvex subgroups of $\pi_1 M$.
- Graphical subgroups of graph products of groups that are coarse median spaces [BR20b].
- Convex cocompact subgroups of mapping class groups [FM02, KL08, Ham05].
- More generally, stable subgroups of HHGs [ABD21].

5.1 The coarse Helly property

We need a lemma from [HHP20] about convex hulls in CAT(0) cube complexes. This will also be used in Section 6.

5.3 Lemma. *Let Q be a CAT(0) cube complex of dimension ν. Given $A \subset Q$, let $A_0 = A$, and set*

$$A_{i+1} = \mu(A_i, A_i, Q) = \{\mu(a, a', v) \ : \ a, a' \in A_i, v \in Q\}.$$

Letting $\nu' = \max\{1, \nu - 1\}$, we have $A_{\nu'} = \operatorname{hull} A$.

Proof. The result is trivial if A is convex. Otherwise, fix $x \in \operatorname{hull}(A) \smallsetminus A$, and let $\mathcal{H}$ be the collection of hyperplanes of $\operatorname{hull} A$ that are adjacent to x. For each $h \in \mathcal{H}$, let $Q = h^+ \sqcup h^-$ denote the partition defined by h, where $x \in h^+$. Let $\{h_1, \dots, h_n\}$ be a maximal pairwise crossing family in $\mathcal{H}$. We have $n \leqslant \nu$. For each i, let $\mathcal{H}_i$ denote the set of elements of $\mathcal{H}$ that are disjoint from h_i, together with h_i. An important observation is that $h_i^- \subset h^+$ whenever $h \in \mathcal{H}_i \smallsetminus \{h_i\}$.

If $n = 1$, then x is a cut-point or leaf of (the graph formed by the 1–skeleton of) $\operatorname{hull} A$, so taking any $a \in A \cap h^+$, $b \in A \cap h^-$ gives $x = \mu(x, a, b)$, and we are done.

So suppose that $n \geqslant 2$. If for every $a \in A \cap h_1^-$ we have $a \in h_2^-$, then for every $b \in A \cap h_2^+$ we have $b \in h_1^+$, so if we take $z_1 \in A \cap h_1^-$ and $z_2 \in A \cap h_2^+$, then $\mu(x, z_1, z_2) \in h^+ \cap h'^+$ for every $h \in \mathcal{H}_1$, $h' \in \mathcal{H}_2$. We can reason similarly if every element of $A \cap h_2^-$ lies in h_1^-. Otherwise there exist $z_1 \in A \cap h_1^- \cap h_2^+$ and $z_2 \in A \cap h_1^+ \cap h_2^-$, and we again have $\mu(x, z_1, z_2) \in h^+ \cap h'^+$ for every $h \in \mathcal{H}_1$, $h' \in \mathcal{H}_2$. Let $y_1 = \mu(x, z_1, z_2) \in A_1$.

We proceed inductively. Suppose that we have $y_i \in A_i$ such that $y_i \in h^+$ for all $h \in \bigcup_{j \leqslant i+1} \mathcal{H}_j$. Let z_{i+2} be any point of A that is separated from y_i by h_{i+2}, and set $y_{i+1} = \mu(x, y_i, z_{i+2})$. Since $x, y_i \in h^+$ for every $h \in \bigcup_{j \leqslant i+1} \mathcal{H}_j$, the same is true of y_{i+1}, and since y_i and z_{i+2} lie on opposite sides of h_{i+2}, we also have that $y_{i+1} \in h^+$ for all $h \in \mathcal{H}_{i+2}$.

By this procedure, we obtain $y_{n-1} \in A_{n-1} \cap \operatorname{hull} A$ that is not separated from x by any hyperplane of $\operatorname{hull} A$, so we must have $y_{n-1} = x$. $\square$

The constant ν' is almost certainly not optimal. A likely candidate seems to be $\lceil \log_2 \nu \rceil$, as this is optimal when A is the star of a vertex in a ν–cube, which one would expect to be the worst case.

We now consider median-quasiconvex subsets of quasicubical coarse median spaces. Recall that a quasiinverse of a quasiisometry $f : Y \to Z$ is a map $\bar{f} : Z \to Y$ such that $\mathsf{d}(y, \bar{f}f(y))$ is uniformly bounded over all $y \in Y$.

5.4 Proposition. *Let X be a quasicubical coarse median space, and let $f : X \to Q$ be a quasimedian quasiisometry, with quasiinverse $\bar{f}$, where Q is a finite-dimensional CAT(0) cube complex. For each k there is an r such that if $Y \subset X$ is k–median-quasiconvex, then the convex subcomplex* $\operatorname{hull} f(Y) \subset Q$ *satisfies* $\mathsf{d}_{Huas}(f(Y), \operatorname{hull} f(Y)) \leqslant r$. *Moreover, there is a constant k_0 such that $\bar{f}(Y')$ is k_0–median-quasiconvex for every convex subcomplex $Y' \subset Q$.*

Proof. In the notation of Lemma 5.3, we have $\operatorname{hull} f(Y) = f(Y)_{\dim Q}$. Using this, it is straightforward to see from median-quasiconvexity of Y and the quasimedian property of f that $\operatorname{hull} f(Y)$ is at bounded Hausdorff-distance from $f(Y)$. The reverse direction is obvious. $\square$

The following is a slight extension of the CAT(0) cube complex version of Helly's theorem (see [Rol98, Thm 2.2]); it is surely well known, but I have been unable to locate it in the literature.

5.5 Lemma (Helly's theorem). *Let Q be a CAT(0) cube complex, and let $\mathcal{Y}$ be a collection of convex subcomplexes of Q such that either $\mathcal{Y}$ is finite or some element of $\mathcal{Y}$ is finite. If each pair of elements of $\mathcal{Y}$ intersects, then $\bigcap_{\mathcal{Y}} Y \neq \varnothing$.*

Proof. Let $Y_0, Y_1, Y_2 \in \mathcal{Y}$, and if $\mathcal{Y}$ has a finite element then ensure that Y_0 is one such. Let $y_{ij} \in Y_i \cap Y_j$. By convexity, $\mu(y_{01}, y_{02}, y_{12}) \in Y_0 \cap Y_1 \cap Y_2$. Repeating this inductively with Y_0, Y_1, and $\bigcap_{i=0}^{\alpha} Y_i$ gives the result if $\mathcal{Y}$ is finite. In the case where $\mathcal{Y}$ is infinite but Y_0 is finite, we use transfinite induction and the fact that a nested sequence of nonempty subsets of a finite set is nonempty. $\square$

5.6 Corollary. *Let X be a quasicubical coarse median space, and let $f : X \to Q$ be a quasimedian quasiisometry, where Q is a finite-dimensional CAT(0) cube complex. For every k and r there is a number R as follows. Suppose that $\mathcal{Y}$ is a collection of k–median-quasiconvex subsets of X that are pairwise r–close. If either $\mathcal{Y}$ is finite or some element of $\mathcal{Y}$ is finite, then there is some $x \in X$ with $\mathsf{d}(x, Y) \leqslant R$ for all $Y \in \mathcal{Y}$.*

Proof. Fix $Y_0 \in \mathcal{Y}$, ensuring that Y_0 is finite if possible. For $Y \in \mathcal{Y}$, let $Y' = \mathcal{N}(Y, r)$ if $Y \neq Y_0$, and let $Y_0' = Y_0$. The Y' are uniformly median-quasiconvex and intersect pairwise. According to Proposition 5.4, the convex subcomplexes $\operatorname{hull} f(Y')$ are at bounded Hausdorff-distance from the $f(Y')$. Moreover, if Y_0 is finite then so is $\operatorname{hull} f(Y_0)$. The $\operatorname{hull} f(Y')$ intersect pairwise, so Lemma 5.5 shows that there is some $v \in \bigcap_{\mathcal{Y}} \operatorname{hull} f(Y')$. The point $x = \bar{f}(v)$ is as desired. $\square$

5.7 Remark. In [HHP20], a version of Corollary 5.6 is proved for all HHSs by using a nice result of Chepoi–Dragan–Vaxès for hyperbolic spaces [CDV17]. Note that not all CAT(0) cube complexes are HHSs, and it is not known whether all HHSs are quasimedian quasiisometric to CAT(0) cube complexes, so it is not entirely clear how the generalities relate to each other. However, both the results from [CDV17] and [HHP20] give more information about the point x.

5.2 Packing

We now describe an application of Corollary 5.6 to bounded packing.

5.8 Definition (Bounded packing). A finite collection $\mathcal{H}$ of subgroups of a discrete group G has *bounded packing* in G if for each n there is a constant r such that for any collection of n distinct cosets of elements of $\mathcal{H}$, at least two are separated by a distance of at least r.

5.9. The bounded packing property for subgroups of finitely generated groups was introduced by Hruska–Wise [HW09] (see also [HW14]) as a metric abstraction of tools used by several authors to prove intersection properties of subgroups of hyperbolic groups [GMRS98, RS99], and in turn as a stepping stone towards ensuring cocompactness of the cube complex associated with a finite collection of quasiconvex codimension–1 subgroups [Sag97, NR03]. The prototypical example is that of a quasiconvex subgroup of a hyperbolic group. That such subgroups have bounded packing was first established by Gitik–Mitra–Rips–Sageev, using compactness of the boundary [GMRS98], and another proof was given by Hruska–Wise, using induction on the *height* of the subgroups [HW09].

More general examples have been provided by Antolín–Mj–Sisto–Taylor, who use induction on height to show that finite collections of *stable subgroups* in any finitely generated group have bounded packing [AMST19]. Stable subgroups were introduced by Durham and Taylor [DT15], and they are always hyperbolic. More generally, *Morse subgroups* were introduced independently by Tran [Tra19] and Genevois [Gen20], and the notion is implicit in earlier work of Sisto [Sis16]. A subgroup is stable if and only if it is hyperbolic and Morse [Tra19, Prop. 4.3]. Notably, Tran proved that any finite collection of Morse subgroups has bounded packing [Tra19, Theorem 1.2], again by using induction on height.

In HHSs, median-quasiconvexity is more general. Indeed, every Morse subgroup of a group that is an HHS is median-quasiconvex (for instance by Lemma 3.8), and, for example, $\mathbf{Z}$ is median-quasiconvex in $\mathbf{Z}^2$ but not Morse.

5.10 Theorem. *If $\mathcal{H}$ is a finite collection of median-quasiconvex subgroups of a colourable HHG G, then $\mathcal{H}$ has bounded packing in G.*

Proof. By Corollary 5.6, any finite collection of cosets of elements of $\mathcal{H}$ that are pairwise r–close must all come R–close to a single point $x \in G$. In other words, they all intersect the R–ball about x. Since distinct cosets of a given subgroup are disjoint and balls in G are finite, this bounds the size of the collection of cosets. $\square$

6 Coarse injectivity

This section is based on joint work with Thomas Haettel and Nima Hoda [HHP20]. However, the arguments presented here take place in slightly different generality. Indeed, we work with quasicubical coarse median spaces, rather than the more general coarse median spaces *with quasicubical intervals* [HHP20, Def. 2.8]. Although the assumption here is less general, it is good enough for our purposes, as we intend to apply the results to colourable HHGs via Theorem 4.20. Making this stronger assumption simplifies the arguments in a couple of places, but the global strategy is the same as in [HHP20].

Throughout this section, (X, μ, d) will denote a coarse median space. For the majority, we shall assume that X is quasicubical. For some arguments in Section 6.2, we shall assume that X is geodesic—the corresponding arguments in [HHP20] only assume it to be *roughly* geodesic, but again, we only intend to apply our results to geodesic spaces. At the end of Section 6.3, we shall need to assume that X is locally finite, in order to apply Corollary 5.6.

Our goal is to construct a new metric σ on X that is coarsely injective, quasiisometric to d, and invariant under median-preserving isometries. This is summarised in Theorem 6.20. The construction is a coarse version of one due to Bowditch, who builds a new metric on median metric spaces and proves that it is injective [Bow20b]; see Remark 6.7.

6.1 Construction of the new metric σ

The following is a coarsification of an idea of Bowditch [Bow20b, §3].

6.1 Definition (Contraction)**.** A *contraction* on a coarse median space X is a 1–quasimedian, $(1,1)$–coarsely Lipschitz map $\phi : X \to \mathbf{R}$.

6.2 Definition (Chain-quotient)**.** A *chain-quotient* on a CAT(0) cube complex Q is a map $\psi : Q \to \mathbf{Z}$ obtained from a chain of hyperplanes $c = (h_i)_{i \in I \cap \mathbf{Z}}$ (Definition 2.9) by setting $\psi(v)$ to be the unique i such that $v \in h^+_{i-1} \cap h^-_i$. We say that ψ is dual to c.

6.3 Remark. Chain-quotients are 1–Lipschitz and median-preserving. They are a special case of the more general *restriction quotients* defined by Caprace–Sageev [CS11].

6.4 New metric. Let X be a coarse median space. For $x, y \in X$, define

$$\sigma(x,y) = \sup\{\phi(x) - \phi(y) \ : \ \phi \text{ is a contraction}\}.$$

Note that since the composition of any contraction with an isometry of $\mathbf{R}$ is also a contraction, this is equivalent to defining $\sigma(x,y) = \sup\{|\phi(x) - \phi(y)| \ : \ \phi \text{ is a contraction}\}$.

6.5 Remark. The article [HHP20] considers so-called *K–contractions* for a constant $K \geqslant 0$, and defines the metric σ for an arbitrary $K > 0$. However, none of the properties we are interested in here require any particular choice of K in order to be valid. Also, as noted in [HHP20, Rem. 2.9], differing choices of K actually give bilipschitz metrics. We have therefore essentially fixed the value $K = 1$ for this section, which simplifies certain constants down the line.

6.6 Lemma. *For any coarse median space, σ is a metric.*

Proof. Symmetry holds because the composition of any contraction with an isometry of $\mathbf{R}$ is also a contraction. Given distinct $x, y \in X$, consider the map ϕ sending x to 1 and $X \smallsetminus \{x\}$ to 0, which is a contraction. Hence $\sigma(x,y) \geqslant \phi(x) - \phi(y) = 1$, so σ separates points. Let $x, y, z \in X$. For every contraction ϕ, we have

$$\sigma(x,z) + \sigma(z,y) \geqslant (\phi(x) - \phi(z)) + (\phi(z) - \phi(y)) = \phi(x) - \phi(y).$$

As this holds for every contraction, we have $\sigma(x,z) + \sigma(z,y) \geqslant \sigma(x,y)$. □

6.7 Remark. Bowditch's work on median spaces [Bow20b] uses 0–contractions in the sense of Remark 6.5 where we use contractions, as he is working in a non-coarse setting. This has the advantage that it produces an *injective* metric (Definition 7.1), rather than a merely coarsely injective metric (Definition 6.16), though it does mean that the above proof that points are separated does not work. In fact, Bowditch obtains this retrospectively after showing that his (*a priori* pseudo)metric is bilipschitz to the initial metric. In the case of CAT(0) cube complexes, Bowditch's metric is the piecewise-ℓ^∞–metric, which is equivalent to counting the length of the longest chain of hyperplanes separating two points. Variations on this have been considered in [Gen20, PSZ22].

6.8 Lemma. *If $g \in \operatorname{Isom}(X, \mathsf{d})$ is median-preserving in the sense that $g\mu(x,y,z) = \mu(gx,gy,gz)$ for all $x,y,z \in X$, then $g \in \operatorname{Isom}(X,\sigma)$.*

Proof. If ϕ is a contraction, then so is ϕg^{-1}, and $\phi g^{-1}(gx) - \phi g^{-1}(gy) = \phi(x) - \phi(y)$. $\square$

Let us now show that σ is quasiisometric to d. The quasicubicality assumption makes the proof here considerably easier than in [HHP20].

6.9 Lemma. *If (X,μ,d) is a coarse median space with a λ–quasimedian-quasiisometry to a finite-dimensional CAT(0) cube complex (Q,μ_Q,d^1_Q), then (X,σ) is quasiisometric to (X,d).*

Proof. Let $x,y \in X$. Since contractions are $(1,1)$–coarsely Lipschitz, we have $\sigma(x,y) \leqslant \mathsf{d}(x,y)+1$.

Now, according to [Bow20b, §7] or [BvdV91, Cor. 2.5], there is a chain-quotient ψ such that $\psi f(x) - \psi f(y) = \mathsf{d}^\infty_Q(f(x),f(y))$. Set $\phi = \frac{1}{\lambda}\psi f$. For any $a,b \in X$, we have

$$|\phi(a)-\phi(b)| \leqslant \frac{1}{\lambda}\,\mathsf{d}^\infty_Q(f(a),f(b)) \leqslant \frac{1}{\lambda}\,\mathsf{d}^1_Q(f(a),f(b)) \leqslant \mathsf{d}(a,b)+1,$$

so ϕ is $(1,1)$–coarsely Lipschitz. Moreover, for any $a,b,c \in X$, we have

$$\begin{aligned}
&\big|\psi f(\mu_{abc}) - \mu_{\mathbf{R}}(\psi fa,\psi fb,\psi fc)\big| \\
&\qquad \leqslant \big|\psi f(\mu_{abc}) - \psi\mu_Q(fa,fb,fc)\big| + \big|\psi\mu_Q(fa,fb,fc) - \mu(\psi fa,\psi fb,\psi fc)\big| \\
&\qquad \leqslant \mathsf{d}^1_Q\left(f(\mu_{abc}),\ \mu_Q(fa,fb,fc)\right)\ \leqslant\ \lambda.
\end{aligned}$$

Hence ϕ is a contraction. In particular, $\sigma(x,y) \geqslant \phi(x)-\phi(y)$, so

$$\begin{aligned}
\sigma(x,y) &\geqslant \frac{1}{\lambda}\,\mathsf{d}^\infty_Q(f(x),f(y)) \\
&\geqslant \frac{1}{\lambda\dim Q}\,\mathsf{d}^1_Q(f(x),f(y)\ \geqslant\ \frac{1}{\lambda\dim Q}\left(\frac{1}{\lambda}\,\mathsf{d}(x,y)-\lambda\right).
\end{aligned}$$

$\square$

6.2 σ is weakly roughly geodesic

Here we show that (X,σ) is *weakly roughly geodesic*, assuming that (X,d) is quasicubical and geodesic. This is a vital part of the proof that (X,σ) is coarsely injective. It is also the most technical part. We follow [HHP20, §2.4]. An outline is provided after a definition.

6.10 Definition (Weakly roughly geodesic)**.** A metric space (X,σ) is *weakly roughly geodesic* if there is a constant C_σ such that for every $a,b \in X$ and every $r \in [0,\sigma(a,b)]$, there is a point $c \in X$ with $|\sigma(a,c)-r| \leqslant C_\sigma$ and $\sigma(a,c)+\sigma(c,b) \leqslant \sigma(a,b)+C_\sigma$.

Weak rough geodesicity does admit a more symmetric formulation, where we require $\sigma(a,c) \leqslant r + C_\sigma$ and $\sigma(c,b) \leqslant \sigma(a,b) - r + C_\sigma$, but the given version will be more convenient for us to verify.

6.11 Overview.

The overall strategy is as follows. Given a pair of points a and b, we wish to find a third point that looks as though it could almost be on a σ–geodesic between them. We use the quasimedian quasiisometry $f : X \to Q$.

As CAT(0) cube complexes are geodesic, we could take a point γ in the correct place along a geodesic from $f(a)$ to $f(b)$. However, this does not *a priori* tell us anything about $\sigma(a, \bar{f}(\gamma))$. Motivated by the fact that the ℓ^∞–metric on Q can be constructed via chain-quotients [Bow20b, BvdV91], we could instead try to use a geodesic in (Q, d^∞).

Taking a point γ along such a geodesic, we then need to compute $\sigma(a, \bar{f}(\gamma))$. We could try to use the fact that d^∞ is characterised in terms of chain-quotients to do this, but it is not clear that all chain-quotients can be pulled back along f to give maps that can be approximated by contractions. We should therefore consider the subset of chain-quotients whose pullbacks *can* be approximated like this, rather than all chain-quotients. This impacts the construction of γ, and the technicalities are dealt with in Lemma 6.13.

This leaves the potential problem that this subset of chain-quotients could be empty. This possibility is disposed of in Lemma 6.12, which shows that for any contraction on X, its pullback along $\bar{f}$ can be approximated by a scaled copy of some chain-quotient.

We are then left to compute $\sigma(a,c) + \sigma(c,b)$. Although Lemmas 6.12 and 6.13 have given us approximable chain-quotients for the pairs $(f(a), \gamma)$ and $(\gamma, f(b))$ that are restrictions of a common chain-quotient, we do not know whether that larger chain-quotient is approximable. We get around this with Lemma 6.14, which provides a way to glue contractions on X. Its proof uses geodesicity of d, but not quasicubicality.

6.12 Lemma. *If Q is a CAT(0) cube complex of dimension ν, then for every contraction χ on Q there is a chain-quotient ψ on Q such that $|\chi - 4\nu\psi| \leqslant 4\nu$.*

Proof. For each $n \in \mathbf{Z}$, let $K_n = \chi^{-1}(4\nu n - 2\nu, 4\nu n]$. Because Q is geodesic, the set of integers n such that $K_n \neq \varnothing$ is an interval $[u-1, v]$. Furthermore, K_n disconnects Q for every $n \in [u, v-1]$.

Claim: If $x \in \operatorname{hull} K_n$, then $\chi(x) \in (4\nu n - 3\nu, 4\nu n + \nu]$.

Proof of Claim. In the notation of Lemma 5.3, consider $x \in (K_n)_i$. According to that lemma, it suffices to show by induction that $|\chi(x) - \chi(K_n)| \leqslant i$. This is clear for $i = 0$. Suppose that it holds for i, and let $x \in (K_n)_{i+1}$. There exist $a_1, a_2 \in (K_n)_i$ such that $x = \mu(x, a_1, a_2)$. Because χ is 1–quasimedian, we have $|\chi(x) - \chi(K_n)| \leqslant 1 + \max\{|\chi(a_j) - \chi(K_n)|\} \leqslant i + 1$. ♢

This means that the convex subcomplexes $\operatorname{hull} K_n$ and $\operatorname{hull} K_m$ are disjoint whenever $n \neq m$. Thus, for each $n \in [u, v]$ there is a hyperplane h_n separating $\operatorname{hull} K_{n-1}$ from $\operatorname{hull} K_n$ [Che94, Cor. 1]. These hyperplanes form a chain, which has a dual chain-quotient $\psi : Q \to [u-1, v]$. Given $x \in Q$, there exists $n \in [u-1, v]$ such that $\chi(x) \in (4\nu n - 4\nu, 4\nu n]$, and we have $\psi(x) \in \{n-1, n\}$. Hence $|\chi(x) - 4\nu\psi(x)| \leqslant 4\nu$. □

Note that Lemma 6.12 applies in particular if we take a contraction $\phi : X \to \mathbf{R}$ on a quasicubical coarse median space and let $\chi = \frac{1}{2\lambda}\phi\bar{f} : Q \to \mathbf{R}$. Thus the pullback of each contraction on X along $\bar{f}$ can be approximated by a chain-quotient. We now show how to modify the idea of "pick a point in the right place along an ℓ^∞–geodesic of Q" when only considering a subset of chain-quotients.

6.13 Lemma. *Let Q be a CAT(0) cube complex, and let $\mathcal{C}$ be a set of chain-quotients on Q. Let $\sigma_{\mathcal{C}}$ be the pseudometric defined by*

$$\sigma_{\mathcal{C}}(\alpha,\beta) \;=\; \max\{|\psi(\alpha)-\psi(\beta)| \;:\; \psi\in\mathcal{C}\} \;\text{ for }\; \alpha,\beta\in Q.$$

For each integer $r \in [0,\sigma_{\mathcal{C}}(\alpha,\beta)]$, there is a vertex $\gamma\in[\alpha,\beta]$ satisfying the following. There are chain-quotients $\psi_1\in\mathcal{C}$, dual to $(h_{1,1},\dots,h_{1,r})$, and $\psi_2\in\mathcal{C}$, dual to $(h_{2,1},\dots,h_{2,s})$, that realise $\sigma_{\mathcal{C}}(\alpha,\gamma)$ and $\sigma_{\mathcal{C}}(\gamma,\beta)$, respectively, and that have the property that $(h_{1,1},\dots,h_{1,r},h_{2,1},\dots,h_{2,s})$ is a chain.

Note that the modulus function can be dropped if $-\psi\in\mathcal{C}$ for every $\psi\in\mathcal{C}$, as will be the case when we apply Lemma 6.13.

Proof of Lemma 6.13. The identity map $(Q,\mathsf{d}^\infty)\to(Q,\sigma_{\mathcal{C}})$ is 1–Lipschitz, so for any $\alpha,\beta\in Q$ and any integer $r\in[0,\sigma_{\mathcal{C}}(\alpha,\beta)]$, there is some $\gamma\in[\alpha,\beta]$ with $\sigma(\alpha,\gamma)=r$. Among all possible choices, let γ be maximally far from α in the sense that

$$\text{if } \gamma'\in[\alpha,\beta] \text{ has } \sigma_{\mathcal{C}}(\alpha,\gamma')=r \text{ and } \gamma\in[\alpha,\gamma'],\ \text{then } \gamma'=\gamma.$$

Let $\psi_2\in\mathcal{C}$ realise $\sigma_{\mathcal{C}}(\gamma,\beta)$, with $(h_{2,1},\dots,h_{2,s})$ the defining hyperplanes separating γ from β. Let h be a hyperplane that is adjacent to γ and either is equal to $h_{2,1}$ or separates γ from $h_{2,1}$. Let $\gamma'\in[\alpha,\beta]$ be the vertex separated from γ by h only. Since $h_{2,1}$ separates γ from β, so must h. As $\gamma\in[\alpha,\beta]$, this means that h cannot separate γ from α, so $\gamma\in[\alpha,\gamma']$. By the choice of γ, we must have $\sigma_{\mathcal{C}}(\alpha,\gamma')=r+1$.

Let $\psi_1\in\mathcal{C}$ realise $\sigma_{\mathcal{C}}(\alpha,\gamma')$, with $(h_{1,1},\dots .h_{1,r+1})$ the defining hyperplanes separating α from γ'. Because only h separates γ' from γ, and because $\sigma_{\mathcal{C}}(\alpha,\gamma)=r$, we know both that $h_{1,r+1}=h$, and that ψ_1 realises $\sigma_{\mathcal{C}}(\alpha,\gamma)$. Moreover, $(h_{1,1},\dots,h_{1,r},h_{2,1},\dots,h_{2,s})$ is a chain, because h is disjoint from $h_{1,r}$, and either $h=h_{2,1}$ or h separates $h_{1,r}$ from $h_{2,1}$. □

Although $(h_{1,1},\dots,h_{1,r},h_{2,1},\dots,h_{2,s})$ in Lemma 6.13 is a chain, the chain-quotient dual to it is not necessarily an element of $\mathcal{C}$. We get around this with the remaining ingredient in the proof of Proposition 6.15: a criterion for gluing together contractions.

6.14 Lemma (Gluing contractions)**.** *Let X be a geodesic coarse median space, let $a,b\in X$, and let $r,s>\varepsilon>0$. Suppose that $\phi_1:X\to[0,r]$ and $\phi_2:X\to[r,r+s]$ are contractions with $\psi_1(a)\leqslant\varepsilon$ and $\phi_2(b)\geqslant r+s-\varepsilon$. If $t\in[0,\min\{r,s\}-\varepsilon-6]$ is such that the sets*

$$Z_1=\{x\in X \;:\; \phi_1(x)\leqslant r-t\} \quad\textit{and}\quad Z_2=\{x\in X \;:\; \phi_2(x)\geqslant r+t\}$$

are disjoint, then $\sigma(a,b)\geqslant r+s-2t-2\varepsilon-14$.

Proof. For $i=0,1$ define $Y_1^i=\{x\in X \;:\; \phi_1(x)\leqslant r-t-7+i\}$, $Y_2^i=\{x\in X \;:\; \phi_2(x)\geqslant r+t+7-i\}$. Note that if $i_1<i_2$ then $Y_j^{i_1}\subset Y_j^{i_2}$.

Claim 1: $\mathsf{d}(Y_1^1,Y_2^1)\geqslant 5$.

Proof of Claim 1. Let $x_j\in Y_j^1$. Since $Y_2^1\subset Z_2$, we have $x_2\notin Z_1$, so $\phi_1(x_2)>r-t$. On the other hand, $\phi_1(x_1)\leqslant r-t-6$, so $\phi_1(x_2)-\phi_1(x_1)\geqslant 6$. As ϕ_1 is $(1,1)$–coarsely Lipschitz, this means that $\mathsf{d}(x_1,x_2)\geqslant 5$. ◇

Let $Y=X\smallsetminus(Y_1^0\cup Y_2^0)$, and define a map $\psi:X\to[0,r+s-2t-14]$ as follows.

$$\begin{aligned}
&\text{If } x\in Y_1^0 \text{ then } \psi(x)=\phi_1(x).\\
&\text{If } x\in Y \text{ then } \psi(x)=r-t-7.\\
&\text{If } x\in Y_2^0 \text{ then } \psi(x)=\phi_2(x)-2t-14.
\end{aligned}$$

We shall show that ψ is a contraction, which will complete the proof because $\psi(a) \leqslant \varepsilon$ and $\psi(b) \geqslant r+s-2t-14-\varepsilon$.

Let $m_1 : [0,r] \to [0, r-t-7]$ be the median-preserving 1–Lipschitz map given by $m_1(z) = \min\{z, r-t-7\}$. Observe that $\psi|_{Y_1^0 \cup Y} = m_1\phi_1$. In particular, $\psi|_{Y_1^0 \cup Y}$ is 1–quasimedian and $(1,1)$–coarsely Lipschitz. Defining m_2 as a similar maximum function shows the same of $\psi|_{Y \cup Y_2^0}$.

Claim 2: ψ is $(1,1)$–coarsely Lipschitz.

Proof of Claim 2. It suffices to consider $x \in Y_1^0$ and $y \in Y_2^0$. Let γ be a geodesic from x to y. There exists τ such that $\gamma(\tau) \in Y_1^0$ but $\gamma(\tau') \notin Y_1^0$ for any $\tau' > \tau$. Write $\gamma(\tau) = z_1$. We have $\mathsf{d}(x,z_1) + \mathsf{d}(z_1,y) = \mathsf{d}(x,y)$ and $\phi_1(z_1) \leqslant r-t-7$. Moreover, for any $\tau' > \tau$, we have

$$\phi_1(z_1) \;\geqslant\; \phi_1\gamma(\tau') - \mathsf{d}(\gamma(\tau),\gamma(\tau')) - 1 \;>\; r-t-7-(\tau'-\tau)-1,$$

so $\phi_1(z_1) \geqslant r-t-9$. We can similarly construct $z_2 \in Y_2^0$ along a geodesic from z_1 to y so that $\phi_2(z_2) \leqslant r+t+9$.

Using these, we compute

$$\begin{aligned}\psi(y)-\psi(x) &\leqslant |\psi(y)-\psi(z_2)| + |\psi(z_2)-\psi(z_1)| + |\psi(z_1)-\psi(x)| \\ &= |\phi_2(y)-\phi_2(z_2)| + |\phi_2(z_2)-2t-14-\phi_1(z_1)| + |\phi_1(z_1)-\phi_1(x)| \\ &\leqslant \big(\mathsf{d}(y,z_2)+1\big) + 4 + \big(\mathsf{d}(z_1,x)+1\big) \\ &= \mathsf{d}(x,z_1) + \mathsf{d}(z_2,y) + 6 \\ &= \mathsf{d}(x,y) - \mathsf{d}(z_1,z_2) + 6.\end{aligned}$$

From Claim 1, we know $\mathsf{d}(z_1,z_2) \geqslant \mathsf{d}(Y_1^1, Y_2^1) \geqslant 5$, and hence $\psi(y)-\psi(x) \leqslant \mathsf{d}(x,y)+1$. $\diamondsuit$

Claim 3: ψ is 1–quasimedian.

Proof of Claim 3. In view of the observation that $\psi|_{Y_1^0 \cup Y} = m_1\phi_1$, and similarly when restricted to $Y \cup Y_2^0$, there are essentially only two cases to consider.

In the first case, suppose that $x, y \in Y_1^0$ and $z \in Y_2^0$. Then $\psi(x), \psi(y) \leqslant r-t-7$, whereas $\psi(z) = \phi_2(z) - 2t - 14 \geqslant r-t-7$. Moreover, $\phi_1(z) > r-t-7$ as $z \notin Y_1^0$. This means that

$$\mu_{\mathbf{R}}\big(\psi(x),\psi(y),\psi(z)\big) = \mu_{\mathbf{R}}\big(\phi_1(x),\phi_1(y),\phi_1(z)\big) \leqslant r-t-7.$$

As ϕ_1 is 1–quasimedian, we get that $\phi_1(\mu_{xyz}) \leqslant r-t-6$, so $\mu_{xyz} \in Y_1^1$, which is disjoint from Y_2^0 by Claim 1. Hence $\psi(\mu_{xyz}) = m_1\phi_1(\mu_{xyz})$, and we can compute

$$\begin{aligned}|\psi(\mu_{xyz}) - \mu_{\mathbf{R}}(\psi x, \psi y, \psi z)| &= |m_1\phi_1(\mu_{xyz}) - \mu_{\mathbf{R}}(\phi_1 x, \phi_1 y, \phi_1 z)| \\ &= |m_1\phi_1(\mu_{xyz}) - \mu_{\mathbf{R}}(m_1\phi_1 x, m_1\phi_1 y, m_1\phi_1 z)| \;\leqslant\; 1.\end{aligned}$$

A similar argument applies when $x \in Y_1^0$ and $y, z \in Y_2^0$.

The remaining case is when $x \in Y_1^0$, $y \in Y$, $z \in Y_2^0$. Since $\psi(x) = \phi_1(x) \leqslant r-t-7$, $\psi(y) = r-t-7$, and $\psi(z) = \phi_2(z) - 2t - 14 \geqslant r-t-7$, we have $\mu_{\mathbf{R}}(\psi(x),\psi(y),\psi(z)) = r-t-7$. If $\mu_{xyz} \in Y$, then $\psi(\mu_{xyz}) = r-t-7$. If $\mu_{xyz} \in Y_1^0$, then $\psi(\mu_{xyz}) = \phi_1(\mu_{xyz}) \leqslant r-t-7$. As ϕ_1 is 1–quasimedian and $y, z \notin Y_1^0$, we also have $\phi_1(\mu_{xyz}) \geqslant r-t-8$, and so $\mu_{\mathbf{R}}(\psi x, \psi y, \psi z) - \psi(\mu_{xyz}) \leqslant 1$. We can argue similarly when $\mu_{xyz} \in Y_2^0$. $\diamondsuit$

Thus ψ is a contraction, so $\sigma(a,b) \geqslant \psi(b) - \psi(a) \geqslant r+s-2\varepsilon-2t-14$. $\square$

We now have all the pieces we need to show that σ is weakly roughly geodesic.

6.15 Proposition. *If* (X, μ, d) *admits a* λ*–quasimedian-quasiisometry to a* ν*–dimensional CAT(0) cube complex* Q*, then* σ *is weakly roughly geodesic.*

Proof. We may assume that $\lambda \geqslant 1$. Given $a, b \in X$, let $\alpha = f(a)$ and $\beta = f(b)$. Lemma 6.9 shows that there is some $q \geqslant 1$ such that σ and d are q–quasiisometric, so $\sigma(a, \bar{f}(\alpha)), \sigma(b, \bar{f}(\beta)) \leqslant 2q\lambda$.

Let $\mathcal{C}$ denote the set of chain-quotients $\psi : Q \to \mathbf{R}$ such that there is some contraction $\phi : X \to \mathbf{R}$ with $|\phi\bar{f} - 8\lambda\nu\psi| \leqslant 8\lambda\nu$. For each contraction $\phi : X \to \mathbf{R}$, the rescaled pullback $\frac{1}{2\lambda}\phi\bar{f} : Q \to \mathbf{R}$ is a contraction on Q, and Lemma 6.12 tells us that there is a chain-quotient $\psi : Q \to \mathbf{R}$ such that $|\phi\bar{f} - 8\lambda\nu| \leqslant 8\lambda\nu$. In other words, every contraction on X gives rise to an element of $\mathcal{C}$.

We shall prove that σ is weakly roughly geodesic with constant

$$C_\sigma = 22\lambda(6\nu + q).$$

Let $r \in [0, \sigma(a, b)]$. If $r \leqslant C_\sigma$, then we can take $c = a$ for the desired point. Similarly, if $r \geqslant \sigma(a, b) - C_\sigma$, then we can take $c = b$. Otherwise, let $r' = \lfloor \frac{r}{8\lambda\nu} \rfloor$ and apply Lemma 6.13 to α, β, $\mathcal{C}$, and r' to obtain a vertex $\gamma \in [\alpha, \beta]$ and chain-quotients ψ_1 and ψ_2. Let $c = \bar{f}(\gamma)$.

Claim 1: $|\sigma(a, c) - r| \leqslant C_\sigma$.

Proof of Claim 1. By definition of $\mathcal{C}$, for any contraction $\phi : X \to \mathbf{R}$ there is some $\psi \in \mathcal{C}$ (and conversely, for any $\psi \in \mathcal{C}$ there is a contraction ϕ) such that

$$\begin{aligned}
&\big||\phi\bar{f}(\xi) - \phi\bar{f}(\zeta)| \;-\; 8\lambda\nu|\psi(\xi) - \psi(\zeta)|\big| \\
&\qquad \leqslant \big||\phi\bar{f}(\xi) - 8\lambda\nu\psi(\xi)| \;-\; |\phi\bar{f}(\zeta) - 8\lambda\nu\psi(\zeta)|\big| \;\leqslant\; 16\lambda\nu
\end{aligned}$$

holds for all $\xi, \zeta \in Q$. It follows that

$$\big|\sigma(\bar{f}(\xi), \bar{f}(\zeta)) - 8\lambda\nu\sigma_{\mathcal{C}}(\xi, \zeta)\big| \;\leqslant\; 16\lambda\nu. \tag{2}$$

By the choice of γ, we have $\sigma_{\mathcal{C}}(\alpha, \gamma) = r'$. Therefore, (2) gives

$$\begin{aligned}
|\sigma(a, c) - r| &\leqslant |\sigma(\bar{f}(\alpha), \bar{f}(\gamma)) - 8\lambda\nu r'| + 2q\lambda + 8\lambda\nu \\
&\leqslant 24\lambda\nu + 2q\lambda,
\end{aligned}$$

which is at most C_σ. ♢

It remains to show that $\sigma(a, c) + \sigma(c, b) \leqslant \sigma(a, b) + C_\sigma$. The strategy is to apply Lemma 6.14.

Recall that Lemma 6.13 has provided chain-quotients ψ_1 and ψ_2 such that $r' = \sigma_{\mathcal{C}}(\alpha, \gamma) = \psi_1(\gamma) - \psi_1(\alpha)$ and $s' = \sigma_{\mathcal{C}}(\gamma, \beta) = \psi_2(\beta) - \psi_2(\gamma)$. After translations of $\mathbf{R}$, we may assume that $\psi_1(\alpha) = 0$, $\psi_1(\gamma) = \psi_2(\gamma) = r'$, and $\psi_2(\beta) = r' + s'$.

By definition of $\mathcal{C}$, there are contractions ϕ_1 and ϕ_2 on X such that $|\phi_i\bar{f} - 8\lambda\nu\psi_i| \leqslant 8\lambda\nu$. In particular, $\phi_1(a) \leqslant \phi_1(\bar{f}(\alpha)) + \lambda + 1 \leqslant 10\lambda\nu$. Moreover, from (2) we obtain

$$\begin{aligned}
\phi_2(b) &\geqslant 8\lambda\nu(r' + s') - 10\lambda\nu \\
&= 8\lambda\nu(\sigma_{\mathcal{C}}(\alpha, \gamma) + \sigma_{\mathcal{C}}(\gamma, \beta)) - 10\lambda\nu \\
&\geqslant \sigma(\bar{f}(\alpha), \bar{f}(\gamma)) + \sigma(\bar{f}(\gamma), \bar{f}(\beta)) - 42\lambda\nu \\
&\geqslant \sigma(a, c) + \sigma(c, b) - 2q\lambda - 42\lambda\nu.
\end{aligned}$$

Furthermore, there is no loss in composing ϕ_1 with $\max\{\cdot, 0\}$ and $\min\{\cdot, r\}$ to assume that $\phi_1 : X \to [0, r]$, and we may similarly assume that $\phi_2 : X \to [r, r+s]$, where $s = \sigma(a,c) + \sigma(c,b) - r$.

Claim 2: The assumptions of Lemma 6.14 are met with $\varepsilon = 42\lambda\nu + 2q\lambda$ if we take $t = 24\lambda\nu + 2q$.

Proof of Claim 2. We must first check that $t \in [0, \min\{r,s\} - \varepsilon - 6]$. Indeed, we have $r - \varepsilon - 6 \geqslant C_\sigma - \varepsilon - 6 \geqslant t$, and also

$$\begin{aligned}\sigma(a,c) + \sigma(c,b) - r - \varepsilon - 6 \ &\geqslant\ \sigma(a,b) - r - \varepsilon - 6 \\ &\geqslant\ C_\sigma - \varepsilon - 6 \ \geqslant\ t.\end{aligned}$$

Now let Z_1, Z_2 be as in the statement of Lemma 6.14. If $x \in Z_1$, then $\phi_1 \bar{f}(fx) \leqslant \phi_1(x) + q + 1 \leqslant r - t + q + 1$, so $\psi_1(fx) \leqslant \frac{r-t+q+1+8\lambda\nu}{8\lambda\nu} \leqslant \frac{r-16\lambda\nu}{8\lambda\nu} \leqslant r' - 1$. A similar argument shows that if $x \in Z_2$, then $\psi_2(fx) \geqslant r' + 1$. The final property of the chains in Lemma 6.13 prohibits these from being satisfied simultaneously. ♢

Applying Lemma 6.14, we find that

$$\begin{aligned}\sigma(a,b) \ &\geqslant\ \sigma(a,c) + \sigma(c,b) - 2t - 2\varepsilon - 14 \\ &=\ \sigma(a,c) + \sigma(c,b) - 132\lambda\nu - 4q - 4q\lambda - 14 \\ &\geqslant\ \sigma(a,c) + \sigma(c,b) - C_\sigma.\end{aligned}$$

Together with Claim 1, this completes the proof. □

6.3 σ is coarsely injective

Here we show that (X, σ) has the following property when X is quasicubical, geodesic, and locally finite.

6.16 Definition (Coarse injectivity)**.** A metric space (X, σ) is *coarsely injective* if there exists ε such that for any collection of balls $B(x_i, r_i)$ with $r_i + r_j \geqslant \sigma(x_i, x_j)$ for all i, j, the total intersection $\bigcap B(x_i, r_i + \varepsilon)$ is nonempty.

Weak rough geodesicity allows us to uniformly thicken such a family of balls so that they intersect pairwise. Our strategy will be to show that balls are median-quasiconvex and apply Corollary 5.6.

6.17 Lemma. *If (x, μ, d) is a coarse median space with a λ–quasimedian-quasiisometry to a CAT(0) cube complex, then $\mathsf{d}(\mu_{abc}, \mu(a, b, \mu_{abc})) \leqslant 6\lambda^2$.*

Proof. Because CAT(0) cube complexes are median spaces, we have

$$\begin{aligned}\mathsf{d}\left(f(\mu_{abc}), f(\mu(a,b,\mu_{abc}))\right) \ &\leqslant\ \mathsf{d}\left(\mu_Q(fa,fb,fc),\, \mu_Q(fa,fb,f\mu_{abc})\right) + 2\lambda \\ &\leqslant\ \mathsf{d}\left(\mu_Q(fa,fb,fc),\, \mu_Q\big(fa,fb,\mu_Q(fa,fb,fc)\big)\right) + 3\lambda \ =\ 3\lambda.\end{aligned}$$

Hence $\mathsf{d}(\mu_{abc}, \mu(a,b,\mu_{abc})) \ \leqslant\ \mathsf{d}\left(\bar{f}f(\mu_{abc}), \bar{f}f(\mu(a,b,\mu_{abc}))\right) + 2\lambda \ \leqslant\ 3\lambda(\lambda+1)$. □

The above distance is actually bounded in all coarse median spaces [Bow19, Lem. 8.1], [HHP20, Lem. 2.22], but restricting like this gives us a more explicit constant.

Let C_σ be such that σ is weakly roughly geodesic with constant C_σ.

6.18 Lemma. *Every ball in (X, σ) is $(6\lambda^2 + 3C_\sigma + 2)$–median-quasiconvex.*

Proof. Fix $a \in X$ and $r > 0$. Let $b, c \in B_\sigma(a, r)$. We want to bound the distance from μ_{bcx} to $B_\sigma(a, r)$ for all $x \in X$. Let ϕ be any contraction with $\phi(\mu_{bcx}) > \phi(a)$. Lemma 6.17 tells us that $\mathsf{d}(\mu_{bcx}, \mu(b, c, \mu_{bcx})) \leqslant 6\lambda^2$, so $\phi(\mu(b, c, \mu_{bcx})) \geqslant \phi(\mu_{bcx}) - 6\lambda^2 - 1$. Thus

$$\begin{aligned} \mu_{\mathbf{R}}(\phi b, \phi c, \phi \mu_{bcx}) &\geqslant \phi(\mu(b, c, \mu_{bcx})) - 1 \\ &\geqslant \phi(\mu_{bcx}) - 6\lambda^2 - 2. \end{aligned}$$

In particular, at least one of $\phi(b)$ and $\phi(c)$ must be at least $\phi(\mu_{bcx}) - 6\lambda^2 - 2$.

As this holds for all contractions ϕ with $\phi(\mu_{bcx}) > \phi(a)$, we have that $\sigma(a, \mu_{bcx}) \leqslant \max\{\sigma(a, b), \sigma(a, c)\} + 6\lambda^2 + 2$. That is, $\mu_{bcx} \in B_\sigma(a, r + 6\lambda^2 + 2)$. By weak rough geodesicity, there is a point $d \in B_\sigma(a, r)$ with $\sigma(d, \mu_{bcx}) \leqslant 6\lambda^2 + 2 + 3C_\sigma$. □

6.19 Proposition. *If (X, μ, d) is a locally finite, quasicubical, geodesic coarse median space, then (X, σ) is coarsely injective.*

Proof. Suppose we have balls $B_\sigma(x_i, r_i)$ with $r_i + r_j \geqslant \sigma(x_i, x_j)$ for all i, j. By Lemma 6.18, they are uniformly median-quasiconvex, and the weak rough geodesicity of σ provided by Proposition 6.15 means that they are uniformly close pairwise. Moreover, the fact that (X, σ) is quasiisometric to the locally finite space (X, d) (Lemma 6.9) shows that every $B_\sigma(x_i, r_i)$ is finite. Corollary 5.6 thus provides a point $x \in X$ and a constant R that is independent of the $B_\sigma(x_i, r_i)$ such that $\sigma(x, B_\sigma(x_i, r_i)) \leqslant R$ for every ball. Hence $\bigcap B_\sigma(x_i, r_i + R) \neq \varnothing$. □

6.4 Summary of properties of σ

The following theorem summarises the various properties of σ that we have proved in the main case of interest.

6.20 Theorem. *If (X, μ, d) is a locally finite, quasicubical, geodesic coarse median space, then it is quasiisometric to the coarsely injective space (X, σ). Moreover, median-preserving isometries of (X, μ, d) are isometries of (X, σ).*

Proof. The fact that (X, σ) is coarsely injective is Proposition 6.19, and the quasiisometry is provided by Lemma 6.9. The statement about isometries is Lemma 6.8. □

6.21 Remark. It follows from coarse injectivity that σ is actually roughly geodesic, rather than merely weakly roughly geodesic, because it is coarsely dense in its injective hull, which is a geodesic space.

We finish this section by making explicit the implication for groups of Theorem 6.20.

6.22 Corollary. *Suppose that G is a finitely generated group that is a quasicubical coarse median space. If the regular action of G is median-preserving, then G acts properly coboundedly on the coarsely injective space (G, σ).*

7 Injective metric spaces

In this section, we discuss some of the theory of injective metric spaces. In Section 8, we shall use this theory together with Theorems 4.20 and 6.20 to deduce consequences for colourable HHGs, some of which are noted in [HHP20]. A good reference for basics of injective spaces is [Lan13].

7.1 Definition (Injectivity)**.** A geodesic metric space is *injective* if every family of pairwise intersecting balls has nonempty total intersection.

7.2 Remark. Clearly every injective space is 0–coarsely injective, and it turns out that every 0–coarsely injective metric space is geodesic, hence injective. On the other hand, there are non-geodesic spaces, such as $\{n^3 : n \in \mathbf{Z}\}$ or $\{(x, |x|)\} \subset (\mathbf{R}^2, \mathsf{d}^1)$, that are not coarsely injective, even though every family of pairwise intersecting balls has nonempty total intersection. See Lemma 7.7 for more information on the relation between the two notions.

Injective spaces were introduced by Aronszajn–Panitchpakdi [AP56], and there are several other characterisations; see [Lan13, §2].

7.1 Injective hulls

It is an interesting and useful fact due to Isbell that every metric space has an essentially unique *injective hull* [Isb64]. This fact was later rediscovered by Dress [Dre84] and Chrobak–Larmore [CL94]. We follow the exposition of [Lan13].

7.3. Let (X, d) be a metric space, and write $\mathbf{R}^X = \{f : X \to \mathbf{R}\}$, which we equip with the (extended) metric $\mathsf{d}^\infty(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. As usual, we write $f \leqslant g$ if $f(x) \leqslant g(x)$ for all $x \in X$, and $f < g$ if $f \leqslant g$ but $f \neq g$. (It is worth pointing out that $f < g$ does *not* mean that $f(x) < g(x)$ for all $x \in X$.) Define the subspace

$$\mathcal{E}(X) = \{f \in \mathbf{R}^X : f(x) + f(y) \geqslant \mathsf{d}(x, y) \text{ for all } x, y \in X\}.$$

Taking $y = x$ in the definition shows that $f \geqslant 0$ for all $f \in \mathcal{E}(X)$.

There are two important ways to construct elements of $\mathcal{E}(X)$. Firstly, for $z \in X$, let $\mathsf{d}_z : X \to \mathbf{R}$ be given by $\mathsf{d}_z(x) = \mathsf{d}(z, x)$. By the triangle inequality, $\mathsf{d}_z \in \mathcal{E}(X)$ for all $z \in X$. Secondly, given $f_1, \dots, f_n \in \mathcal{E}(X)$ and $t_1, \dots, t_n \in \mathbf{R}$ with $\sum t_i \geqslant 1$, the sum $\sum t_i f_i \in \mathcal{E}(X)$. In particular, the affine segment joining any two points of $\mathcal{E}(X)$ is contained in $\mathcal{E}(X)$.

7.4 Injective hull. Call an element $f \in \mathcal{E}(X)$ *minimal* if it is minimal in $\mathcal{E}(X)$ with respect to $<$. If $f \in \mathbf{R}^X$ and there is some $x \in X$ for which $f(x) < \sup\{\mathsf{d}(x, y) - f(y) : y \in X\}$, then there is some $y \in X$ with $\mathsf{d}(x, y) - f(y) > f(x)$, whence $f \notin \mathcal{E}(X)$. On the other hand, if $f(x) > \sup\{\mathsf{d}(x, y) - f(y) : y \in X\}$, then there is some positive δ such that $f(x) + f(y) > \mathsf{d}(x, y) + \delta$ for all y, so f is not minimal. This shows that:

- $f \in \mathcal{E}(X)$ if and only if $f(x) \geqslant \sup\{\mathsf{d}(x, y) - f(y) : y \in X\}$ for all $x \in X$, and
- $f \in \mathcal{E}(X)$ is minimal if and only if $f(x) = \sup\{\mathsf{d}(x, y) - f(y) : y \in X\}$ for all $x \in X$.

As an example, we see that every d_z is minimal. Let

$$E(X) = \{f \in \mathcal{E}(X) : f \text{ is minimal}\},$$

and define a map $e : X \to E(X)$ by $e : z \mapsto \mathsf{d}_z$. It is straightforward to see that e is an isometric embedding.

7.5 Theorem ([Isb64]). *$E(X)$ is injective. Any isometric embedding of X in an injective space factors via e. In particular, $E(X) = e(X)$ if X is injective.*

We call $E(X)$ the *injective hull* of X.

7.6 Isometries. The action of $\operatorname{Isom} X$ on X gives rise to an action of $\operatorname{Isom} X$ on $\mathcal{E}(X)$. Namely, for any $\phi \in \operatorname{Isom} X$ and $f \in \mathcal{E}(X)$, the map $\phi \cdot f = f\phi^{-1} : X \to \mathbf{R}$ is an element of $\mathcal{E}(X)$. Moreover, this restricts to an action on $E(X)$. Although it would appear to be more natural to define this as a right action, it will be seen from the continued discussion that the left action is actually more convenient.

As an example, we compute $\phi \cdot \mathsf{d}_z = \mathsf{d}(z, \phi^{-1}x) = \mathsf{d}(\phi z, x) = \mathsf{d}_{\phi z}(x)$. That is, $\phi \cdot \mathsf{d}_z = \mathsf{d}_{\phi z}$ for all $z \in X$. Equivalently, if we write $\hat{\phi}$ for the image of ϕ in $\operatorname{Sym}(\mathcal{E}(X))$, then we have

$$\hat{\phi} e = e\phi. \tag{3}$$

It can be shown from an alternative characterisation of injectivity that $\hat{\phi}$ is the unique isometry of $E(X)$ that satisfies this equality; see [Lan13, Prop. 3.7].

The following lemma, whose proof is identical to to that of [CCG⁺20, Prop. 3.12], shows the connection between coarsely injective spaces and injective spaces.

7.7 Lemma. *A metric space is coarsely injective if and only if it is coarsely dense in its injective hull. In particular, if a group G acts properly coboundedly on a coarsely injective space X, then G acts properly coboundedly on $E(X)$.*

7.8 A projection map. We now describe a retraction $p : \mathcal{E}(X) \to E(X)$ that interacts nicely with e and $\operatorname{Isom} X$. This retraction was originally constructed by Dress [Dre89].

For $f \in \mathcal{E}(X)$, define $f^*(x) = \sup\{\mathsf{d}(x,y) - f(y) \ : \ y \in X\}$. Note that $f \in E(X)$ if and only if $f^* = f$. In any case, $f^* \leqslant f$. Although f^* might not lie in $\mathcal{E}(X)$, the function $q(f) = \frac{1}{2}(f + f^*)$ *is* an element of $\mathcal{E}(X)$. Moreover, for any $f, g \in \mathcal{E}(X)$, we have

$$f^*(x) \ = \ \sup\{\mathsf{d}(x,y) - g(y) + g(y) - f(y) \ : \ y \in X\} \ \leqslant \ g^*(x) + \mathsf{d}^\infty(f,g),$$

so $\mathsf{d}^\infty(f^*, g^*) \leqslant \mathsf{d}^\infty(f,g)$, and hence q is 1–Lipschitz.

Define $p(f)$ to be the pointwise limit of the sequence $(q^n(f))_n$.

7.9 Proposition ([Dre89]). *p is a 1–Lipschitz retraction $\mathcal{E}(X) \to E(X)$, and $\hat{\phi} p = p\hat{\phi}$ for all $\phi \in \operatorname{Isom} X$.*

Proof. To check that p is a 1–Lipschitz retraction to $E(X)$, it remains only to check that the codomain is correct. For all $f \in \mathcal{E}(X)$ and all $n \geqslant 1$, we have $p(f) \leqslant q^n(f)$, and hence $p(f)^* \geqslant q^n(f)^*$. But now

$$\begin{aligned} 0 \ \leqslant \ p(f) - p(f)^* \ \leqslant \ q^n(f) - q^n(f)^* \ &= \ 2q^n(f) - (q^n(f) + q^n(f)^*) \\ &= \ 2(q^n(f) - q^{n+1}(f)), \end{aligned}$$

which converges pointwise to 0 as $n \to \infty$. This shows that $p(f) \in E(X)$. It is easily seen that $\hat{\phi}(q(f)) = q(\hat{\phi}(f))$ for all $f \in \mathcal{E}(X)$ and $\phi \in \operatorname{Isom} X$, and it follows that $\hat{\phi} p = p\hat{\phi}$. $\square$

7.2 Barycentric spans

Let X be an injective metric space. Here we give a new construction, which we call the *barycentric span* of a finite subset of X. By specialising this construction, we can obtain both the bicombing of [Lan13, Prop. 3.8] and a barycentre map.

Write Δ^n for the standard n–dimensional simplex. That is, $\Delta^n = \{(s_1, \ldots, s_{n+1}) \in [0,1]^{n+1} : \sum s_i = 1\}$. Given a finite ordered list of elements $a_1, \ldots, a_n$ of a set A, we shall write $\tilde{a} = (a_1, \ldots, a_n) \in A^n$.

7.10 Definition (Barycentric span)**.** Given a finite tuple $\tilde{x} = (x_1, \ldots, x_n) \in X^n$ and a point $\tilde{s} \in \Delta^{n-1}$, let

$$\Delta_{\tilde{x}}(\tilde{s}) \;=\; e^{-1}p\big(\sum s_i\,\mathsf{d}_{x_i}\big) \;\in\; X.$$

The *barycentric span* of points $x_1, \ldots, x_n \in X$ is the subspace $\Delta_{\tilde{x}} = \{\Delta_{\tilde{x}}(\tilde{s}) : \tilde{s} \in \Delta^{n-1}\}$.

Recall that for a tuple $\tilde{a} = (a_1, \ldots, a_n) \in A^n$ and a permutation $\sigma \in S_n$, if $a_{\sigma(i)} = a_i$ for all i, then $\sigma\tilde{a} = \tilde{a}$, so $\sigma \in \operatorname{Stab}\tilde{a}$.

7.11 Proposition. *Let X be an injective space. Barycentric spans are*

1. permutation-invariant *in the sense that $\Delta_{\sigma\tilde{x}}(\sigma\tilde{s}) = \Delta_{\tilde{x}}(\tilde{s})$ for any $\sigma \in S_n$;*
2. equivariant *in the sense that $\phi\Delta_{\tilde{x}}(\tilde{s}) = \Delta_{\phi\tilde{x}}(\tilde{s})$ for any $\phi \in \operatorname{Isom} X$;*
3. metrically stable under perturbations *in the sense that*

$$\mathsf{d}(\Delta_{\tilde{x}}(\tilde{s}), \Delta_{\tilde{y}}(\tilde{s})) \;\leqslant\; \min\big\{\sum s_i\,\mathsf{d}(x_i, y_{\sigma(i)}) \;:\; \sigma \in \operatorname{Stab}\tilde{s}\big\}$$

for any $\tilde{x}, \tilde{y} \in X^n$.

Proof. Item 1. is clear from the definition. Item 2. holds because of the compatibility of e and p with isometries given in Equation (3) and Proposition 7.9. Item 3. holds because e is an isometry and p is 1–Lipschitz. □

7.12 Isoperimetry. One can use barycentric spans to efficiently fill cycles in X, which shows that any group G acting properly coboundedly on X has at most Euclidean k^{th}-*order isoperimetric functions*, i.e. $O(n^{\frac{k+1}{k}})$, for all k. In particular, G has at most quadratic Dehn function and is of type F_∞. As we shall see in Section 7.3, injective spaces have *conical bicombings*, and this also implies the bounds on the higher Dehn functions [ECH$^+$92, Thm 10.2.1] (also see [BD19]). A more explicit Dehn function is given using conical bicombings in [Cre20].

We illustrate the procedure by sketching a proof of the following lemma, which implies the statement about quadratic Dehn function.

7.13 Lemma. *Suppose that $\tilde{x} = (x_0, \ldots, x_{n-1}, x_n = x_0) \in X^n$ satisfies $\mathsf{d}(x_{i-1}, x_i) \leqslant r$ for all i. There is a triangulation of the disc D with vertex set V and at most $(2n-1)(n-2)$ 2–cells such that there is a map $f : V \to X$ that sends the boundary cycle of D to $\tilde{x}$ and has the property that $\mathsf{d}(f(v_1), f(v_2)) \leqslant 2r$ whenever $v_1, v_2 \in V$ are adjacent.*

Sketch proof. Let ∇ be the affine simplex in $\mathcal{E}(X)$ spanned by $\mathsf{d}_{x_0}, \ldots, \mathsf{d}_{x_{n-1}}$. Let D be the disc formed from the union of the 2–cells $(x_0, x_i, x_{i+1}) \subset \nabla$, of which there are $n-2$. Each of these affine 2–cells can be subdivided into $2n-1$ smaller 2–cells whose vertices are pairwise $2r$–close, which gives V. The map $f = e^{-1}p|_V$ is 1–Lipschitz. □

7.3 Bicombings

The notions of combings and bicombings originated in the idea of considering only metric aspects from the theory of automaticity [ECH$^+$92, Alo92, Sho90]. Since bicombings were introduced, their scope has expanded considerably [AB95, DL15, EW17].

7.14 Definition (Bicombing)**.** A *bicombing* γ on a metric space X is a choice of constant-speed geodesic $\gamma_{x,y} : [0,1] \to X$ from x to y for each pair $(x,y) \in X^2$.

We now list a few additional properties that a bicombing may have.

7.15 Definition. A bicombing γ on a metric space (X, d) is

- *reversible* if $\gamma_{x,y} = \gamma_{y,x}$ for all $x, y \in X$;
- *G–equivariant* if $g\gamma_{x,y} = \gamma_{gx,gy}$ for all $x, y \in X$, $g \in G$, where $G < \operatorname{Isom} X$;
- *consistent* if $\gamma_{z,w} \subset \gamma_{x,y}$ whenever $z, w \in \gamma_{x,y}$;
- *conical* if for every $x, x', y, y' \in X$ and every $t \in [0,1]$ we have

$$\mathsf{d}(\gamma_{x,y}(t), \gamma_{x',y'}(t)) \leqslant (1-t)\,\mathsf{d}(x,x') + t\,\mathsf{d}(y,y').$$

If a bicombing is consistent and conical, then the map $t \mapsto \mathsf{d}(\gamma_{x,y}(t), \gamma_{x',y'}(t))$ is convex for all x, x', y, y', as is the case for the unique geodesics in a CAT(0) space [Bal95, Prop. 5.4].

Let us now specialise barycentric spans by considering them for a pair of points. In this case $\Delta_{(x,y)}$ is a path for each $x, y \in X$. Let $\gamma_{x,y} : t \mapsto \Delta_{(x,y)}((1-t,t))$. As the 1–Lipschitz image of the affine geodesic $[\mathsf{d}_x, \mathsf{d}_y] \subset \mathcal{E}(X)$, which has length $\mathsf{d}(x,y)$, the path $\gamma_{x,y}$ is a geodesic.

We therefore have a bicombing on injective spaces X; this bicombing was first described in [Lan13, Prop. 3.8]. The following is immediate from Proposition 7.11.

7.16 Lemma. *The above bicombing γ is reversible, equivariant, and conical.*

7.17 Remark. It follows from the existence of γ that injective spaces are contractible, but this can be more simply seen from an alternative characterisation of injectivity [Isb64, Thm 1.1].

7.18 Inconsistency. It turns out that γ can fail to be consistent. For example, let X be the injective hull of the five points $a = (0,4)$, $b = (1,3)$, $(3,5)$, $(4,0)$, and $c = (6,2)$ in $(\mathbf{R}^2, \mathsf{d}^\infty)$; this is depicted in Figure 7. One can easily compute that $\gamma_{b,c}$ is the affine geodesic, which passes through $(2, \frac{14}{5})$, whereas $\gamma_{a,c}(\frac{1}{3}) = (2, \frac{8}{3})$.

Rescaled copies of this example can be embedded in the Cayley complex of $\mathbf{Z}^2 * \mathbf{Z}$ (also known as the *tree of flats*) where the flats are given the ℓ^∞–metric. This shows that, for this space, the bicombing γ is not even at a bounded distance from being consistent, even though the space has a proper cocompact group action.

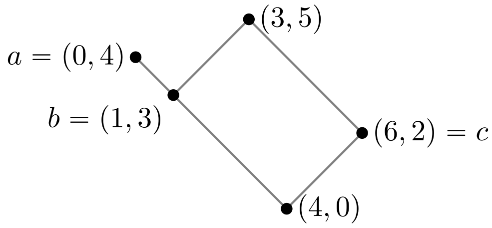

Figure 7: An injective subspace of $(\mathbf{R}^2, \mathsf{d}^\infty)$ where the bicombing γ is not consistent.

7.19 Remark. Although the bicombing γ can fail to be consistent, Descombes–Lang showed that every proper injective space possesses a unique reversible, consistent, conical bicombing [DL15, Thms 1.1, 1.2]. Unfortunately, it does not seem to be a simple matter to determine whether the injective hull of a given metric space is proper. For example, the three metric spaces $X_0 = (\mathbf{R}^n, \mathsf{d}^\infty)$, $X_1 = (\mathbf{R}^n, \mathsf{d}^1)$, and $X_2 = (\mathbf{R}^n, \mathsf{d}^2)$ are pairwise bilipschitz, but X_0 is injective, $E(X_1) = (\mathbf{R}^{2^{n-1}}, \mathsf{d}^\infty)$ [Her92], and $E(X_2)$ has infinite topological dimension.

In a similar vein, note that Lemma 7.7 is most naturally a metric statement, rather than a topological one, as a cocompact action on a coarsely injective space will not give rise to a cocompact action on the injective hull when that hull is not proper. It would therefore be desirable to have a theory of groups acting properly coboundedly on non-proper injective spaces, especially in view of Corollary 6.22.

7.4 Barycentres

Here we specialise the barycentric span in another way to obtain a barycentre map on the injective space X.

7.20 Definition (Barycentre)**.** Given $\tilde{x} = (x_1, \dots, x_n) \in X^n$, the *barycentre* is defined to be $b_{\tilde{x}} = b(x_1, \dots, x_n) = \Delta_{\tilde{x}}(\frac{1}{n}, \dots, \frac{1}{n}) \in X$.

The following is immediate from Proposition 7.11.

7.21 Lemma. *Barycentres are invariant under permutations of their defining points. If $\phi \in \operatorname{Isom} X$, then $\phi b(x_1, \dots, x_n) = b(\phi x_1, \dots, \phi x_n)$. The barycentre map is Lipschitz in its coordinates, in that $\mathsf{d}(b(x_1, \dots, x_n), b(y_1, \dots, y_n)) \leqslant \frac{1}{n} \sum \mathsf{d}(x_i, y_i)$.*

7.22. Es-Sahib–Heinich [ESH99] introduced a barycentre map for *Busemann spaces*, which was then reviewed and partially improved by Navas [Nav13]. It was later observed by Descombes [Des16] that the construction works equally well for complete metric spaces with reversible conical bicombings; this includes all injective spaces. However, since the most natural construction of such a bicombing on an injective space is via taking the barycentric span of two points, that route to a barycentre is somewhat circuitous.

Moreover, it was shown by Basso [Bas20] that if Y is a metric space with a reversible conical bicombing γ, then there is a conical bicombing on $E(Y)$ whose restriction to Y is γ.

Recall that the translation length of $\phi \in \operatorname{Isom} Y$ is $|\phi| = \inf\{\mathsf{d}(y, \phi y) : y \in Y\}$, and $\operatorname{Min} \phi = \{y \in Y : \mathsf{d}(y, \phi y) = |\phi|\}$. We say ϕ is *hyperbolic* if $|\phi| > 0$ and $\operatorname{Min} \phi \neq \varnothing$. The next couple of lemmas should be compared to [Des15, §7].

7.23 Lemma. *If ϕ is a hyperbolic isometry of X, then $\operatorname{Min} \phi$ is closed under taking barycentric spans, and ϕ has a geodesic axis.*

Proof. The closure is immediate from Proposition 7.11. This shows that if $x \in \operatorname{Min} \phi$, then the bicombing geodesic $\gamma_{x,\phi x}$ is contained in $\operatorname{Min} \phi$. The concatenation $\bigcup_{n \in \mathbf{Z}} \phi^n \gamma_{x,\phi x}$ is a geodesic axis for ϕ. ☐

7.24 Lemma. *If $\phi \in \operatorname{Isom} X$ and some ϕ^n is hyperbolic, then ϕ is hyperbolic.*

Proof. Fix $x \in \operatorname{Min} \phi^n$, and let $\tilde{x} = (x, \phi x, \dots, \phi^{n-1} x)$. Since ϕ and ϕ^n commute, every $\phi^k x$ lies in $\operatorname{Min} \phi^n$, so Lemma 7.23 tells us that $b_{\tilde{x}} \in \operatorname{Min} \phi^n$. By the triangle inequality, $\mathsf{d}(b_{\tilde{x}}, \phi b_{\tilde{x}}) \geqslant \frac{1}{n}|\phi^n|$. On the other hand, Proposition 7.11 shows that

$$\mathsf{d}(b_{\tilde{x}}, \phi b_{\tilde{x}}) \;=\; \mathsf{d}(b(x, \phi x, \dots, \phi^{n-1} x),\, b(\phi^n x, \phi x, \dots, \phi^{n-1} x)) \;\leqslant\; \frac{1}{n}\, \mathsf{d}(x, \phi^n x) \;=\; \frac{1}{n}|\phi^n|.$$

Thus ϕ is hyperbolic and $b_{\tilde{x}} \in \operatorname{Min} \phi$. $\square$

7.25 Remark. A similar argument shows that $|\phi| = \lim \frac{1}{n} \mathsf{d}(x, \phi^n x)$ for all $x \in X$.

In [KMV21, §5], Keppeler–Möller–Varghese use the statement of Lemma 7.24 to show that $\mathbf{Q}$ cannot be a subgroup of any group that acts properly cocompactly on an injective space.

7.26 Lemma. *If F is a finite subgroup of* $\operatorname{Isom} X$*, then F has a fixed point.*

Proof. Write $F = \{1, f_2, \ldots, f_n\}$ and let $x \in X$. The barycentre $b(x, f_2 x, \ldots, f_n x)$ is fixed by F because of Lemma 7.21. $\square$

A stronger version of Lemma 7.26 was proved by Lang [Lan13, Prop. 1.2]. Either form has a standard consequence [BH99, Prop. I.8.5].

7.27 Corollary. *If G acts properly coboundedly on an injective space X, then G has finitely many conjugacy classes of finite subgroups.*

Proof. Let $x \in X$ and let r be such that $G \cdot x$ is r–dense in X. If F is a finite subgroup of G, then F fixes some point $z \in X$ by Lemma 7.26, so F fixes the ball $B(z, r)$, which contains a point of $G \cdot x$. Thus some conjugate of F fixes a point in $B(x, r)$, so we are done by properness of the action. $\square$

7.5 Groups acting on injective spaces

Let us summarise the results of this section in the presence of a proper cobounded group action.

7.28 Theorem. *Suppose that a group G acts properly coboundedly on an injective space X.*

- *G has at most Euclidean k^{th}-order isoperimetric functions for all k.*
- *X has a reversible, conical, G–equivariant bicombing.*
- *X has a permutation-invariant, G–equivariant barycentre map that is Lipschitz in its coordinates.*
- *$\mathbf{Q}$ is not a subgroup of G [KMV21, Prop. 5.3].*
- *G has finitely many conjugacy classes of finite subgroups.*

8 Conclusions

In this final section, we pull together the results of the previous sections to obtain facts about colourable HHGs.

8.1 Theorem. *If G is a colourable HHG, then G acts properly coboundedly on the coarsely injective space (G,σ), where σ is as defined in Section 6.1.*

Proof. G is a coarse median space by Proposition 3.7, and, as noted in Remark 3.15, the median can be chosen so that G acts on itself by median-preserving isometries. Also, G is locally finite, roughly geodesic, and, according to Theorem 4.20, it is quasicubical. By Corollary 6.22, G acts properly coboundedly on the coarsely injective space (G,σ). □

8.2 Corollary. *Let G be a colourable HHG.*

- *G has at most Euclidean k^{th}-order isoperimetric functions for all k, so is of type F_∞.*
- *G is semihyperbolic (Definition 8.4). In particular,*
  - *G has soluble conjugacy problem.*
  - *All polycyclic subgroups of G are virtually abelian.*
  - *All finitely generated abelian subgroups of G are undistorted.*
- *For any ring R we have* $\operatorname{cohomdim}_R(G) \leqslant 1 + \operatorname{asdim} G$.
- *G has a contracting barycentre map in the sense of [DMS20, Def. 6.2].*
- $\mathbf{Q}$ *is not a subgroup of G.*
- *G has finitely many conjugacy classes of finite subgroups.*

Proof. According to Theorem 8.1, G acts properly coboundedly on the coarsely injective space (G,σ), hence on the injective space $E(G,\sigma)$ by Lemma 7.7. The consequences follow from Theorem 7.28, as we now describe.

Semihyperbolicity holds because the bicombing in Theorem 7.28 is *bounded* in the sense of [AB95] (see Definition 8.4. The bicombing is also *coherent* and *expanding* in the sense of [EW17], so the bound on cohomological dimension is given by [EW17, Thm C]. The fact that $\mathbf{Q}$ is not a subgroup is [KMV21, Prop. 5.3]. □

The class of colourable HHGs is a proper subclass of the class of groups admitting a proper cobounded action on an injective space. Indeed, non-colourable HHGs such as the one described in [Hag21] also admit such actions [HHP20], and there are also uniform lattices that act properly coboundedly on injective spaces but are not hierarchically hyperbolic [CCHO20, CCG$^+$20, Hae20].

The remainder of the section consists of a discussion of the consequences in Corollary 8.2.

8.3 F_∞. Recall that all HHGs are coarse median spaces of finite rank. A nice argument of Bowditch shows that such groups have at most quadratic Dehn function [Bow13, Prop. 8.2]. Also, their *asymptotic cones* are bilipschitz to median metric spaces [Bow13, §9], and hence are bilipschitz both to injective spaces [Bow20b] and to CAT(0) spaces [Bow16b]. In particular, the asymptotic cones of any HHG are contractible [BHS19], so such groups are of type F_∞ by a theorem of Riley [Ril03, Thm D].

Semihyperbolicity was introduced by Alonso–Bridson [AB95] in response to a call by Gromov for a weakened version of hyperbolicity [Gro87] and as part of the exploration of automaticity-related properties.

8.4 Definition (Semihyperbolic)**.** A *bounded quasigeodesic bicombing* β on a metric space X is a choice of uniform quasigeodesic $\beta_{x,y}$ from x to y for each pair $(x, y) \in X^2$ such that there is a constant k for which

$$\mathsf{d}(\beta_{x,y}(t), \beta_{x',y'}(t)) \leqslant k \max\{\mathsf{d}(x, x'), \mathsf{d}(y, y')\} + k$$

for all $x, x', y, y' \in X$. A group G is *semihyperbolic* if there is a space X with a bounded quasigeodesic bicombing β and a proper cobounded G–action such that $g\beta_{x,y} = \beta_{gx,gy}$ for all $g \in G$ and all $x, y \in X$ [AB95, Thm 4.1].

The following is a special case of Corollary 8.2. It was deduced in this way in [HHP20], and simultaneously proved by Durham–Minsky–Sisto using very different methods [DMS20] (but again as a special case of a more general result; see Section 1.4).

8.5 Theorem. *Mapping class groups are semihyperbolic.*

This complements Mosher's theorem that mapping class groups are automatic [Mos95], as these are the two components of *biautomaticity*, which requires knowing that the bicombing provided by a generating set witnessing automaticity is itself bounded and equivariant.A proof of biautomaticity of mapping class groups appears in unpublished work of Hamenstädt [Ham09]. On the other hand, recent examples of Hughes–Valiunas show that not all colourable HHGs are biautomatic [HV22].

8.6 Conjugacy problem. The result about the conjugacy problem extends work of Abbott–Behrstock showing that it can be solved in exponential time for *Morse elements* of HHGs [AB18], and generalises the fact that it can always be solved in exponential time in mapping class groups [MM00, Tao13, BD14].

8.7 Other consequences. The result about polycyclic subgroups of HHGs can also be deduced from the Tits alternative for HHGs [DHS17, DHS20]. Undistortion of finitely generated abelian subgroups of HHGs was also proved by Plummer[1]. A barycentre with similar properties is produced for colourable HHGs in [DMS20]; it has the added benefit of *equivariance*, and it always lies in the median–quasiconvex hull. The fact that mapping class groups have finitely many conjugacy classes of finite subgroups is also known as a consequence of Kerckhoff's solution of the Nielsen realisation problem [Ker83, Bri00].

[1]Private communication.

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